

A Comparison of two Models of Orbispaces

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This exposition proves that the two homotopy theories for orbispaces given by Gepner and Henriques and by Schwede, respectively, agree by providing a zig-zag of Dwyer-Kan equivalences between the respective topologically enriched index categories. The aforementioned authors establish various models for unstable global homotopy theory with compact Lie group isotropy, and orbispaces serve as a common denominator for their particular approaches. Although the two flavors of orbispaces are expected to agree with each other, a concrete comparison zig-zag has not been known so far. We bridge this gap by providing such a zig-zag which asserts that all those models for unstable global homotopy theory with compact Lie group isotropy which have been described by the authors named above agree with each other.

On our way, we provide a result which is of independent interest. For a large class of free actions of a compact Lie group, we prove that the homotopy quotient by the group action is weakly equivalent to the strict quotient. This is a known result under more restrictive conditions, e.g., for free actions on a manifold. We broadly extend these results to all free actions of a compact Lie group on a compactly generated Hausdorff space.

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1. Introduction

Global homotopy theory is concerned with *global spaces* where all groups act upon at the same time. A map between two global spaces is considered to be a *global equivalence* if it induces a weak equivalence on all fixed points.

In the above, “all groups” refers to a fixed *global family* of groups, i.e., a class of groups subject to certain conditions. Consequently, “all fixed points” means fixed points with respect to those groups in the chosen global family. Popular choices of global families are the family of all finite groups or the family of all compact Lie groups. This paper will only be concerned with the latter.

Note that there is also a stable version that studies global spectra up to global equivalence. In contrast, the version outlined in the previous paragraphs is commonly known as *unstable global homotopy theory*. In this paper, we are concerned with the unstable case only and will therefore omit this adjective frequently.

To sum things up, “global” means “unstable and global with respect to compact Lie groups” for the remainder of this paper.

It remains to make the notion of a global space more precise. Various models have emerged: In [4], Gepner and Henriques equip stacks and topological groupoids with certain homotopical structures and prove that they yield homotopy theories that are equivalent to their version of orbispaces. This gives rise to three different models for global spaces. All of these models do actually work for more general settings, namely for certain *families of allowed isotropy groups* consisting of topological groups.

In [12], Schwede introduces \mathcal{L} -spaces and orthogonal spaces along with his version of orbispaces, together with model category structures on all of these categories, and verifies that they are Quillen equivalent. Therefore, he also describes three models for global spaces. These models rely on properties specific to compact Lie groups and cannot be easily adapted to more general settings.

In both cases, orbispaces are space-valued enriched presheaves on a topologically enriched small category, the *orbit category*, equipped with the projective model structure. However, the two variants of orbispaces mentioned above are not equal on the nose because the respective orbit categories differ.

As we will recall, there is significant evidence that the two versions of orbispaces should be equivalent. Given two compact Lie groups H, G , the mapping spaces $\mathbf{O}_{\text{gl}}(H, G)$ and $\text{Orb}(H, G)$ of the two different orbit categories have the same weak homotopy type, see Remark 2.11, but, at least to the author’s knowledge, no concrete zig-zag of maps has been written down so far.

This exposition shows that there is a zig-zag of weak equivalences between the respective mapping spaces which is compatible with the given structures of topologically enriched categories. We deduce that there is a zig-zag of Dwyer-Kan equivalences between the orbit category \mathbf{O}_{gl} from [12] and the orbit category Orb from [4]. This implies that the associated presheaf categories are Quillen equivalent via a zig-zag, and as a consequence, all the models for global homotopy theory from both papers are equivalent to each other.

The author would like to acknowledge that the main idea for the construction of the zig-zag of Dwyer-Kan equivalences is due to Thomas Nikolaus. He also provided helpful hints and remarks concerning proof strategies and technical details.

1.1. Results

For two completely universal (see Definition 2.6) Lie groups $G, H \subseteq \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$, set

$$\mathbf{O}_{\text{gl}}(H, G) := (\mathbb{L}(\mathbb{R}_G^\infty, \mathbb{R}_H^\infty)/G)^H$$

where \mathbb{L} denotes the space of isometric linear embeddings and $\mathbb{R}_G^\infty, \mathbb{R}_H^\infty$ are both just \mathbb{R}^∞ equipped with the canonical G - and H -action respectively. Moreover, let

$$\text{Orb}(H, G) := \text{map}(H, G) \times_G EG$$

where $\text{map}(H, G)$ denotes the space of continuous (and hence smooth) group homomorphisms from H to G .

These are the mapping spaces of the topologically enriched categories \mathbf{O}_{gl} and Orb which share the same set of objects, namely $\text{ob } \mathbf{O}_{\text{gl}} = \text{ob } \text{Orb} = \{G \mid G \text{ a completely universal Lie group}\}$.

Theorem. *There is a zig-zag of Dwyer-Kan equivalences between \mathbf{O}_{gl} and Orb .*

This appears as Corollary 3.13.

Corollary. *There is a zig-zag of Quillen equivalences between the projective model structures on the enriched presheaf categories $\text{Pre}(\mathbf{O}_{\text{gl}}, \text{Top})$ and $\text{Pre}(\text{Orb}, \text{Top})$.*

This follows immediately using Theorem A.25. The first model category, $\text{Pre}(\mathbf{O}_{\text{gl}}, \text{Top})$ is the variant of orbispaces used in [12] while the second one, $\text{Pre}(\text{Orb}, \text{Top})$ is used in [4].

1.2. Organization of the Paper

The promised zig-zag of Dwyer-Kan equivalences between \mathbf{O}_{gl} and Orb will consist of a third topologically enriched category Orb' with $\text{ob Orb}' = \text{ob } \mathbf{O}_{\text{gl}} = \text{ob Orb}$.

Section 2 focuses on the mapping space $\text{Orb}'(H, G)$ for two fixed completely universal Lie groups H, G and its relation to the mapping spaces $\mathbf{O}_{\text{gl}}(H, G)$ and $\text{Orb}(H, G)$. While Subsection 2.1 recalls the definition of $\text{Orb}(H, G)$ along with auxiliary results, Subsection 2.2 proceeds similarly for $\mathbf{O}_{\text{gl}}(H, G)$.

In Subsection 2.3, we construct a free G -space $\tilde{E}(H, G)$, see Definition 2.12, and set $\text{Orb}'(H, G) := \tilde{E}(H, G) \times_G EG$. This space admits a canonical map $\text{Orb}'(H, G) \rightarrow \text{Orb}(H, G)$, which is readily seen to be a weak equivalence, see Proposition 2.18.

Therefore, it remains to specify a weak equivalence $\text{Orb}'(H, G) \rightarrow \mathbf{O}_{\text{gl}}(H, G)$, which is the content of Subsection 2.4. We show that the quotient space $\text{Orb}'(H, G)/G$ is canonically homeomorphic to $\mathbf{O}_{\text{gl}}(H, G)$, see Proposition 2.20. After showing that the G -action on $\tilde{E}(H, G)$ is free, we deduce that the natural map $\text{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \rightarrow \tilde{E}(H, G)/G \cong \mathbf{O}_{\text{gl}}(H, G)$ is a weak equivalence (Proposition 2.22). This concludes the construction of the zig-zag on the level of individual mapping spaces:

$$\text{Orb}(H, G) \xleftarrow{\simeq} \tilde{E}(H, G) \times_G EG = \text{Orb}'(H, G) \xrightarrow{\simeq} \tilde{E}(H, G)/G \cong \mathbf{O}_{\text{gl}}(H, G).$$

The goal of Section 3 is the extension of this zig-zag to a zig-zag of Dwyer-Kan equivalences. However, the construction of the topologically enriched category $\text{Orb}'(H, G)$ is not complete since we have not specified composition laws yet. This is amended in Subsection 3.1: As the composition law on Orb' is similar to the one on Orb , we simultaneously recall the composition on Orb and define the one on Orb' . In the end, we conclude that the maps $\text{Orb}'(H, G) \rightarrow \text{Orb}(H, G)$ from Subsection 2.3 extend to a Dwyer-Kan equivalence $\text{Orb}' \rightarrow \text{Orb}$. Finally, it is left to verify that the composition law on \mathbf{O}_{gl} is compatible with the maps $\text{Orb}'(H, G) \rightarrow \mathbf{O}_{\text{gl}}(H, G)$ introduced in Subsection 2.4, which is the content of Subsection 3.2. In conclusion, we deduce that there is a zig-zag of Dwyer-Kan equivalences:

$$\text{Orb} \xleftarrow{\simeq} \text{Orb}' \xrightarrow{\simeq} \mathbf{O}_{\text{gl}}$$

From the various auxiliary results of the appendix, we would like to highlight two. First, Theorem A.8 proves that whenever a compact Lie group G acts on a Hausdorff space X , the canonical map

$$X \times_G EG \rightarrow X/G$$

is a weak equivalence. Secondly, Subsection A.7 provides a detailed proof of the well-known fact that a Dwyer-Kan equivalence $\mathcal{C} \simeq \mathcal{D}$ of topologically enriched categories gives rise to a Quillen equivalence $\text{Pre}(\mathcal{C}, \text{Top}) \simeq \text{Pre}(\mathcal{D}, \text{Top})$ between the associated enriched presheaf categories endowed with the respective projective model structures.

2. The Intermediate Mapping Space

Throughout the main body of this paper, the term *space* will always refer to a compactly generated weak Hausdorff space. This convention is explained in more detail in Subsection A.1 of the appendix.

2.1. A Reminder on $\text{Orb}(H, G)$

Definition 2.1. Let H, G be Lie groups.

- (i) Denote by $\text{map}(H, G)$ the space of all continuous (hence smooth) group homomorphism. It comes with a continuous G -action from the right by conjugation, i.e., for $\alpha : H \rightarrow G$ and $g \in G$, define $(\alpha \cdot g)(h) := g^{-1}\alpha(h)g$.

- (ii) The quotient of $\text{map}(H, G)$ by the (usually non-free) G -action is called $\text{Rep}(H, G)$.
- (iii) Using the bar construction as a model for EG , we set

$$\text{Orb}(H, G) := \text{map}(H, G) \times_G EG := |n \mapsto \text{map}(H, G) \times G^n|$$

as the realization of a simplicial topological space.

Remark 2.2. *Gepner and Henriques define $\text{Orb}(H, G)$ as the fat geometric realization of the simplicial topological space $\{n \mapsto \text{map}(H, G) \times G^n\}$, which is weakly equivalent to our definition. We prefer the non-fat realization as it behaves better from a technical point of view for our purposes because it commutes with products on the nose. We will elaborate on the consequences in Remark 3.10.*

In preparation for later results, we derive an important property of the topology on $\text{map}(H, G)$.

Lemma 2.3 (“Close Maps are Conjugate”, [3, Lemma 38.1]). *Let H be a compact Lie group and G a Lie group. For each $\alpha \in \text{map}(H, G)$, there is an open neighborhood $U \subseteq H \times G$ of $\text{graph}(\alpha)$ such that for each $\beta \in \text{map}(H, G)$ with $\text{graph}(\beta) \subseteq U$, β is conjugate to α .*

Proposition 2.4. (i) *Let H be a compact Lie group and G a Lie group. For each $\alpha \in \text{map}(H, G)$, there is an open neighborhood V of α in $\text{map}(H, G)$ such that each $\beta \in V$ is conjugate to α .*

(ii) *The quotient topology on $\text{Rep}(H, G) = \text{map}(H, G)/G$ is discrete.*

(iii) *All G -orbits αG are open. In particular, there is a G -equivariant decomposition*

$$\text{map}(H, G) \cong \coprod_{i \in I} \alpha_i G$$

where $\{\alpha_i\}_{i \in I}$ is a choice of representatives of equivalence classes in $\text{Rep}(H, G)$,

Proof. (i) Pick a metric d_G on G . As H is compact, the topology on $\text{map}(H, G)$ coincides with the one induced by the metric d which is defined as follows:

$$d(\alpha, \beta) := \max_{h \in H} d_G(\alpha(h), \beta(h)).$$

Fix $\alpha \in \text{map}(H, G)$ and choose an open U as in the previous Lemma. For each $h \in H$, pick an open neighborhood $U_h = A_h \times B_{\varepsilon(h)}(\alpha(h))$ of $(h, \alpha(h))$ inside U where $B_{\varepsilon}(\alpha(h))$ denotes an ε -ball around $\alpha(h)$ with respect to d_G . By replacing A_h with $A_h \cap \alpha^{-1}(B_{\varepsilon(h)/2}(\alpha(h)))$, we may assume without loss of generality that $\alpha(A_h) \subseteq B_{\varepsilon(h)/2}(\alpha(h))$.

The A_h cover the compact space H . Therefore, we find a finite set $H_0 \subseteq H$ such that $\{A_h \mid h \in H_0\}$ covers H . Set $\varepsilon := \min_{h \in H_0} \varepsilon(h)$ and denote by

$$V = B_{\varepsilon/2}(\alpha) = \{\beta \in \text{map}(H, G) \mid d(\alpha, \beta) = \max_{h \in H} d_G(\alpha(h), \beta(h)) < \varepsilon/2\}$$

the $\varepsilon/2$ -ball around α in $\text{map}(H, G)$.

Pick $\beta \in V$. We claim that $\text{graph}(\beta) \subseteq U$. Let $h \in H$, then $h \in A_{h'}$ for some $h' \in H_0$. Also, $d_G(\alpha(h), \beta(h)) < \varepsilon/2 < \varepsilon(h')/2$ because $\beta \in V$. As $\alpha(A_{h'}) \subseteq B_{\varepsilon(h')/2}(\alpha(h'))$, we obtain $d_G(\alpha(h'), \alpha(h)) < \varepsilon(h')/2$. The triangle inequality yields

$$d_G(\alpha(h'), \beta(h)) \leq d_G(\alpha(h'), \alpha(h)) + d_G(\alpha(h), \beta(h)) < \varepsilon(h')/2 + \varepsilon(h')/2 = \varepsilon(h').$$

Therefore, $(h, \beta(h)) \in A_{h'} \times B_{\varepsilon(h')/2}(\alpha(h')) \subseteq U$. In conclusion, $\text{graph}(\beta) \subseteq U$ which implies that β is conjugate to α by the choice of U .

- (ii) The quotient projection $\text{map}(H, G) \longrightarrow \text{Rep}(H, G) = \text{map}(H, G)/G$ is open. For $\alpha \in \text{map}(H, G)$, an open neighborhood V as in the previous part is sent to the singleton $\{[\alpha]\}$. Hence, all singletons are open in $\text{Rep}(H, G)$.

- (iii) As each $[\alpha] \in \text{Rep}(H, G)$ is open, so is its preimage αG . The αG are compact, too, as they are the images of the compact sets $\{\alpha\} \times G$ under the action map $\text{map}(H, G) \times G \longrightarrow \text{map}(H, G)$. Since $\text{map}(H, G)$ is metrizable, it is Hausdorff and the compact sets αG must be closed. Choosing $\{\alpha_i\}_{i \in I}$ as in the statement of this proposition, this implies that the canonical map

$$\coprod_{i \in I} \alpha_i G \longrightarrow \text{map}(H, G)$$

is a homeomorphism. The G -action on $\text{map}(H, G)$ restricts to G -actions on the orbits $\alpha_i G$ and the canonical map is G -equivariant with respect to these actions. \square

Of course, the above decomposition always exists on the set level. The new insight is its compatibility with the topology on $\text{map}(H, G)$. This allows us to determine the weak homotopy type of $\text{Orb}(H, G)$:

Proposition 2.5. *Let $\{\alpha_i\}_{i \in I}$ be a choice of representatives of equivalence classes in $\text{Rep}(H, G)$. Then the weak homotopy type of $\text{Orb}(H, G) = \text{map}(H, G) \times_G EG$ is*

$$\coprod_{i \in I} BC(\alpha_i)$$

where $C(\alpha_i) \subseteq G$ denotes the centralizer subgroup of the image of $\alpha_i : H \longrightarrow G$.

Proof. We follow the argumentation in [10, Remark 2.2.1]: By Proposition 2.4.(iii), we have a G -equivariant decomposition $\text{map}(H, G) \cong \coprod_{i \in I} \alpha_i G$. As the functor $- \times_G EG$ commutes with coproducts in the category of G -spaces, we obtain

$$\text{Orb}(H, G) = \text{map}(H, G) \times_G EG \cong \coprod_{i \in I} (\alpha_i G \times_G EG),$$

and it is left to verify that $\alpha_i G \times_G EG$ is a $BC(\alpha_i)$. Note that $\alpha_i G \cong C(\alpha_i) \backslash G$, because $C(\alpha_i)$ is precisely the stabilizer of α_i with respect to the G -action on $\text{map}(H, G)$.

The realization of the simplicial space EG_\bullet , given by $EG_n = G \times G^n$, is the space EG . It has a free G -action which is the realization of the G -action on EG_\bullet through the first factor of the product. The restriction of this action to $C(\alpha_i) \subseteq G$ is free, so $C(\alpha_i) \backslash EG$ is a $BC(\alpha_i)$.

We claim that $C(\alpha_i) \backslash EG$ is exactly $\alpha_i G \times_G EG = C(\alpha_i) \backslash G \times_G EG$. Restricting the G -action on EG_\bullet to $C(\alpha_i)$ yields a $C(\alpha_i)$ -action on EG_\bullet . The quotient by this action is the simplicial topological space with n -th level $C(\alpha_i) \backslash G \times G^n$ whose realization is precisely $C(\alpha_i) \backslash G \times_G EG$. Taking quotients by $C(\alpha_i)$ -actions commutes with realization, so

$$C(\alpha_i) \backslash G \times_G EG = |n \mapsto C(\alpha_i) \backslash G \times G^n| = |C(\alpha_i) \backslash EG_\bullet| \cong C(\alpha_i) \backslash |EG_\bullet| = C(\alpha_i) \backslash EG.$$

Therefore, $C(\alpha_i) \backslash G \times_G EG$ is a $BC(\alpha_i)$, concluding the proof. \square

2.2. A Reminder on $\mathbf{O}_{\text{gl}}(H, G)$

We briefly recall some definitions from [12]:

Definition 2.6. (i) For real inner product spaces V, W of finite or countably infinite dimension, $\mathbb{L}(V, W)$ is the space of linear isometric embeddings topologized as in [12, Section 2].

In the special case $V = W = \mathbb{R}^\infty$, we abbreviate $\mathcal{L} := \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ and observe that composition endows \mathcal{L} with the structure of a unital topological monoid.

- (ii) A compact subgroup $G \subseteq \mathcal{L}$ is *completely universal* [12, Definition 2.10] if

- a) it admits a necessarily unique structure of a Lie group,
- b) the induced orthogonal G -representation on \mathbb{R}^∞ , denoted by \mathbb{R}_G^∞ , is a complete G -universe, i.e., it has non-trivial G -fixed points, and every finite-dimensional G -representation V embeds G -equivariantly into \mathbb{R}^∞ countably many times.

We will refer to completely universal subgroups of \mathcal{L} as *completely universal Lie groups*.

- (iii) Given completely universal Lie groups H, G , we denote by $E(H, G) = \mathbb{L}(\mathbb{R}_G^\infty, \mathbb{R}_H^\infty)$ the right $H \times G$ -space with underlying space $\mathcal{L} = \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ and

$$(f \cdot (h, g))(x) := h^{-1} \cdot f(gx)$$

for $f \in \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$, $(h, g) \in H \times G$, and $x \in \mathbb{R}^\infty$, using the G - and H -actions on \mathbb{R}^∞ .

Finally [12, Construction 2.25],

$$\mathbf{O}_{\text{gl}}(H, G) := (\mathbb{L}(\mathbb{R}_G^\infty, \mathbb{R}_H^\infty)/G)^H = (E(H, G)/G)^H.$$

Remark 2.7. (i) The concept of completely universal Lie groups from Definition 2.6 encompasses all compact Lie groups in the sense that isomorphism classes of compact Lie groups are in bijection with conjugacy classes of completely universal Lie groups [12, Proposition 2.11.].

- (ii) An alternative definition of $\mathbf{O}_{\text{gl}}(H, G)$ can be given by the space of \mathcal{L} -equivariant maps

$$\text{map}_{\mathcal{L}}(\mathcal{L}/H, \mathcal{L}/G).$$

The evaluation at $[\text{id}_{\mathbb{R}^\infty}] \in \mathcal{L}/H$ induces a homeomorphism from this space to our definition of $\mathbf{O}_{\text{gl}}(H, G)$, see also [12, Construction 1.13].

The crucial property of $E(H, G)$, allowing for an identification of the weak homotopy type of $\mathbf{O}_{\text{gl}}(H, G)$, is its universality with respect to *graph subgroups*.

Definition 2.8. Let H, G be compact Lie groups.

- (i) The family of *graph subgroups* ([11, Definition I.2.8]) is defined to be

$$\mathcal{F}(H, G) := \{\text{graph}(\alpha) \subseteq H \times G \mid \alpha : L \longrightarrow G \text{ a Lie group homomorphism for some closed } L \leq H\}.$$

- (ii) If E is an $H \times G$ -space and $\alpha : L \longrightarrow G$, $L \leq H$ closed, then we set $E^\alpha := E^{\text{graph}(\alpha)}$ and, as in Proposition 2.5, we let $C(\alpha) \subseteq G$ denote the centralizer of the image of α .

Proposition 2.9 ([12, Proposition 2.9.(ii)]). *Given completely universal Lie groups H, G , the $H \times G$ -space $E(H, G)$ is a universal space for the family $\mathcal{F}(H, G)$, i.e., for $K \leq H \times G$,*

$$E(H, G)^K \simeq \begin{cases} * & \text{if } K \in \mathcal{F}(H, G) \\ \emptyset & \text{if } K \notin \mathcal{F}(H, G). \end{cases}$$

Proposition 2.10 ([11, Proposition I.5.14.(i)]). *For any choice of representatives $\{\alpha_i\}_{i \in I}$ of equivalence classes in $\text{Rep}(H, G) = \text{map}(H, G)/G$, the canonical map*

$$\lambda : \coprod_{i \in I} E(H, G)^{\alpha_i} / C(\alpha_i) \longrightarrow (E(H, G)/G)^H$$

is well-defined and a homeomorphism. Furthermore, $E(H, G)^{\alpha_i} / C(\alpha_i)$ is a $BC(\alpha_i)$.

Remark 2.11. (i) Therefore, the weak homotopy type of $\mathbf{O}_{\text{gl}}(H, G)$ is $\coprod_{i \in I} BC(\alpha_i)$ (see also [12, Remark 2.27]). This agrees with the weak homotopy type of $\text{Orb}(H, G)$ by Proposition 2.5.

- (ii) Of course, the map λ from before depends on the groups H, G and should be called $\lambda_{H, G}$. However, in order to reduce notational clutter, we chose to only decorate those maps with subscripts that will be considered again for varying groups H, G as we discuss the composition law in the next section (such as the map $\tilde{p}_{H, G}$ from Definition 2.12). Henceforth, this principle will be applied without further remarks.

2.3. The Intermediate Mapping Space $\text{Orb}'(H, G)$

In order to compare $\text{Orb}(H, G)$ to $\mathbf{O}_{\text{gl}}(H, G)$, we introduce a fattened up version of $\text{map}(H, G)$ that incorporates $E(H, G)$. Its homotopy quotient will be $\text{Orb}'(H, G)$.

Convention. For the remainder of this section, let H and G be completely universal Lie groups unless otherwise stated.

Definition 2.12. The G -space $\tilde{E}(H, G)$ has underlying set

$$\{(\alpha, x) \in \text{map}(H, G) \times E(H, G) \mid x \in E(H, G)^\alpha\}$$

topologized as a subspace of $\text{map}(H, G) \times E(H, G)$ and equipped with the diagonal action. It comes with a canonical projection map $\tilde{p}_{H, G} : \tilde{E}(H, G) \rightarrow \text{map}(H, G)$.

Remark 2.13. Lemma A.18 will show that this is a CGWH space.

Lemma 2.14. The G -action on $\tilde{E}(H, G)$ is well-defined.

Proof. Let $\alpha \in \text{map}(H, G)$ and $x \in E(H, G)^\alpha$. We need to show that $xg \in E(H, G)^{\alpha g}$. Spelling out the definitions, this amounts to verifying that for each $h \in H$, the point $xg = x \cdot (1, g)$ is fixed under the action of $(h, g^{-1}\alpha(h)g)$. We compute

$$x \cdot (1, g) \cdot (h, g^{-1}\alpha(h)g) = x \cdot (h, \alpha(h)g) = x \cdot (h, \alpha(h)) \cdot (1, g) = x \cdot (1, g) = xg.$$

The G -action is automatically continuous because it is the restriction of the continuous diagonal action on $\text{map}(H, G) \times E(H, G)$. \square

Proposition 2.15. The map $\tilde{p}_{H, G}$ is a fiber bundle.

Proof. Let us write $\tilde{p} = \tilde{p}_{H, G}$ within this proof. By Proposition 2.4, it suffices to show that the restriction

$$\tilde{p}^{-1}(\alpha G) \rightarrow \alpha G$$

is a fiber bundle for any $\alpha \in \text{map}(H, G)$, where $\alpha G \subseteq \text{map}(H, G)$ is the G -orbit of α .

The canonical map $\text{Stab}(\alpha) \backslash G \rightarrow \alpha G$ is a continuous bijection from a compact space to $\alpha G \subseteq \text{map}(H, G)$. Since the space $\text{map}(H, G)$ is metrizable, it is Hausdorff and so is its subspace αG . Thus, the map is a homeomorphism. Therefore, the map $\tilde{p}^{-1}(\alpha G) \rightarrow \alpha G$ from before is homeomorphic to the projection map

$$\tilde{p}_\alpha : \{(\bar{g}, x) \in (\text{Stab}(\alpha) \backslash G) \times E(H, G) \mid x \in E(H, G)^{\alpha g}\} \rightarrow \text{Stab}(\alpha) \backslash G.$$

Fix a point $\bar{g} \in \text{Stab}(\alpha) \backslash G$. The stabilizer $\text{Stab}(\alpha) \subseteq G$ is a closed subgroup of the Lie group G . In particular, the quotient map $\pi : G \rightarrow \text{Stab}(\alpha) \backslash G$ has local sections and we can choose a neighborhood U of \bar{g} and a section $s : U \rightarrow G$. We are now in the position to write down a local trivialization of \tilde{p}_α :

$$\begin{aligned} \tilde{p}_\alpha^{-1}(U) = \{(\bar{h}, x) \in U \times E(H, G) \mid x \in E(H, G)^{\alpha h}\} &\cong U \times E(H, G)^\alpha \\ (\bar{h}, x) &\mapsto (\bar{h}, x \cdot s(\bar{h})^{-1}) \\ (\bar{h}, x \cdot s(\bar{h})) &\leftarrow (\bar{h}, x) \end{aligned}$$

The maps are continuous because the section s is continuous and it is easy to see that they are mutually inverse. It is left to verify that they are well-defined. For $(\bar{h}, x) \in U \times E(H, G)^\alpha$, we have to check that $x \cdot s(\bar{h}) \in E(H, G)^{\alpha h}$. As s is a section, we can write $s(\bar{h}) = th$ for some $t \in \text{Stab}(\alpha)$. Hence, $x \cdot s(\bar{h}) \in E(H, G)^{\alpha th} = E(H, G)^{\alpha h}$. Well-definedness for the other map works in the same fashion. Finally, it is evident that the homeomorphisms are compatible with the projections to U . Therefore, they constitute a local trivialization of \tilde{p}_α , concluding the proof. \square

Corollary 2.16. The map $\tilde{p}_{H, G}$ is a weak equivalence.

Proof. By the previous proposition, $\tilde{p}_{H,G}$ is a Serre fibration. Over any point $\alpha \in \text{map}(H, G)$, its fiber is exactly $E(H, G)^\alpha$, which is contractible by Proposition 2.9. Consequently, $\tilde{p}_{H,G}$ is a weak equivalence. \square

Definition 2.17. Using the bar construction as a model for EG , we define

$$\text{Orb}'(H, G) := \tilde{E}(H, G) \times_G EG.$$

The map $\tilde{p}_{H,G} : \tilde{E}(H, G) \longrightarrow \text{map}(H, G)$ from Definition 2.12 induces a map

$$l_{H,G} : \text{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \longrightarrow \text{map}(H, G) \times_G EG = \text{Orb}(H, G).$$

by passing to homotopy quotients

Proposition 2.18. *The map $l_{H,G}$ is a weak equivalence.*

Proof. The weak equivalence $\tilde{p}_{H,G} : \tilde{E}(H, G) \longrightarrow \text{map}(H, G)$ induces a levelwise weak equivalence

$$\tilde{E}(H, G) \times G^n \longrightarrow \text{map}(H, G) \times G^n$$

between those simplicial topological spaces that compute $\tilde{E}(H, G) \times_G EG$ and $\text{map}(H, G) \times_G EG$ respectively.

The unit map $* \longrightarrow G$ is a Hurewicz cofibration because G is a smooth manifold. Therefore, any degeneracy map of the two simplicial topological spaces is a Hurewicz cofibration, and both simplicial topological spaces are good. Goodness implies properness, and levelwise weak equivalences between proper simplicial topological spaces realize to weak equivalences. Hence, the induced map $l_{H,G}$ on realizations is a weak equivalence. \square

2.4. The Map to $\mathbf{O}_{\text{gl}}(H, G)$

To complete our desired zig-zag of mapping spaces, we still have to construct a map from $\text{Orb}'(H, G)$ to $\mathbf{O}_{\text{gl}}(H, G)$. At the end of this subsection, we will have proven that $\tilde{E}(H, G)/G = \mathbf{O}_{\text{gl}}(H, G)$ so that we can use the canonical map

$$\text{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \longrightarrow \tilde{E}(H, G)/G = \mathbf{O}_{\text{gl}}(H, G).$$

Proposition 2.19. *Fix a set $\{\alpha_i\}_{i \in I}$ of representatives of $\text{map}(H, G)/G = \text{Rep}(H, G)$. Then the canonical map*

$$\mu : \coprod_{i \in I} E(H, G)^{\alpha_i} / C(\alpha_i) \longrightarrow \tilde{E}(H, G)/G = \{(\alpha, x) \in \text{map}(H, G) \times E(H, G) \mid x \in E^\alpha\} / G,$$

given by its components $\mu_i([x]) = [\alpha_i, x]$, is a homeomorphism.

Proof. We have to show that the components μ_i are well-defined, implying continuity for μ , and that μ is bijective with continuous inverse.

For well-definedness, observe that $C(\alpha_i) = \text{Stab}(\alpha_i)$ with respect to the action of G on $\text{map}(H, G)$. Pick $g \in \text{Stab}(\alpha_i)$ and $x \in E(H, G)^{\alpha_i}$. Then $\mu_i([xg]) = [\alpha_i, xg] = [\alpha_i g, xg] = \mu_i([x])$. Thus μ_i is well-defined, and μ is automatically continuous.

In order to prove surjectivity, pick $[\alpha, x]$ in the codomain of μ . We have $\alpha = \alpha_i g$ for some $i \in I$ and $g \in G$. So, $[\alpha, x] = [\alpha_i g, x] = [\alpha_i, xg^{-1}] = \mu_i([xg^{-1}])$ is in the image of μ .

Considering injectivity, suppose that for $i, j \in I$, we have $x \in E(H, G)^{\alpha_i}$ and $y \in E(H, G)^{\alpha_j}$ such that $\mu_i([x]) = \mu_j([y])$. This means that $[\alpha_i, x] = [\alpha_j, y]$ so there is $g \in G$ such that $(\alpha_i, x) = (\alpha_j g, yg) \in \tilde{E}(H, G)$. In particular, α_i and α_j are in the same orbit and we must have $i = j$. Furthermore, $\alpha_i = \alpha_i g$ implies that $g \in \text{Stab}(\alpha_i)$. As we have $x = yg$, we obtain $[x] = [y] \in E(H, G)^{\alpha_i} / \text{Stab}(\alpha_i) = E(H, G)^{\alpha_i} / C(\alpha_i)$.

It remains to show that the inverse map is continuous. To this end, consider the G -equivariant decomposition $\text{map}(H, G) \cong \coprod_i \alpha_i G$, see Proposition 2.4. Each component $\alpha_i G$ is canonically homeomorphic to $\text{Stab}(\alpha_i) \backslash G$. Set

$$\tilde{E}(H, G)_i := \{(\bar{g}, x) \in \text{Stab}(\alpha_i) \backslash G \times E(H, G) \mid x \in E^{\alpha_i g}\}.$$

This is a G -space via the diagonal action, and we have $\tilde{E}(H, G) \cong \coprod_i \tilde{E}(H, G)_i$ equivariantly. As coproducts commute with quotients, the codomain of μ is equivariantly a coproduct of the $\tilde{E}(H, G)_i / G$ with respect to the maps

$$\begin{array}{ccc} \iota_i : \tilde{E}(H, G)_i / G & \longrightarrow & \tilde{E}(H, G) / G \\ [\bar{g}, x] & \mapsto & [\alpha_i g, x]. \end{array}$$

A computation reveals that the set-theoretic inverse of μ has components

$$\begin{array}{ccc} \xi_i : \tilde{E}(H, G)_i / G & \longrightarrow & E(H, G)^{\alpha_i} / \text{Stab}(\alpha_i) \\ [\bar{g}, x] & \mapsto & [xg^{-1}] \end{array}$$

with respect to the equivariant coproduct decomposition of $\tilde{E}(H, G)/G$ from before. To conclude the proof, we need to show that the ξ_i are continuous. This will be achieved by identifying their domains $\tilde{E}(H, G)_i/G$ as certain quotients.

Fix $i \in I$. Since the fixed points $E(H, G)^{\alpha_i}$ are closed in $E(H, G)$, the subset $G \times E(H, G)^{\alpha_i} \subseteq G \times E(H, G)$ is closed as well. The map $\varrho : G \times E(H, G) \longrightarrow G \times E(H, G)$, $(g, x) \mapsto (g, xg^{-1})$, is continuous. Hence, $A := \varrho^{-1}(G \times E(H, G)^{\alpha_i})$ must be closed. The set A can easily be identified as

$$A = \{(g, x) \in G \times E(H, G) \mid x \in E(H, G)^{\alpha_i g}\}.$$

Next, let us consider the canonical quotient map $p : G \times E(H, G) \longrightarrow (\text{Stab}(\alpha_i) \backslash G) \times E(H, G)$. The subset A of $G \times E(H, G)$ is p -saturated, i.e., $p^{-1}(p(A)) = A$. To see this, we must show that $(g, x) \in A$ implies $(gh, xh) \in A$ for any $h \in G$. But from $x \in E(H, G)^{\alpha_i g}$, it follows that $xh \in E(H, G)^{\alpha_i gh}$ and $(gh, xh) \in A$. Now, since A is closed and saturated, the quotient map p restricts to a quotient map $p|_A : A \longrightarrow p(A)$. Moreover, $p(A) = \tilde{E}(H, G)_i$. In the commutative diagram

$$\begin{array}{ccc} (g, x) & \xrightarrow{\quad} & [x \cdot g^{-1}] \\ \cap & & \cap \\ A & \xrightarrow{\quad} & E(H, G)^{\alpha_i} / \text{Stab}(\alpha_i) \\ \downarrow & \nearrow & \uparrow \\ p(A) & & \xi_i \\ \downarrow & & \uparrow \\ p(A)/G & & \end{array}$$

the vertical maps are quotient maps, and the horizontal map is continuous. Therefore, ξ_i is continuous, and μ is a homeomorphism because its set-theoretic inverse is the coproduct of the continuous maps ξ_i . \square

Proposition 2.20. Define $\nu_{H, G} : \tilde{E}(H, G)/G \longrightarrow (E(H, G)/G)^H$ by the formula $\nu_{H, G}([\alpha, x]) = [x]$. Then $\nu_{H, G}$ is a well-defined homeomorphism and makes the following diagram commutes

$$\begin{array}{ccc} & & \tilde{E}(H, G)/G \\ & \nearrow \mu & \downarrow \nu_{H, G} \\ \coprod_{i \in I} E(H, G)^{\alpha_i} / C(\alpha_i) & & (E(H, G)/G)^H = \mathbf{O}_{\text{gl}}(H, G) \\ & \searrow \lambda & \end{array}$$

where λ is as in Proposition 2.10.

Proof. The map μ is a homeomorphism by Proposition 2.19, and λ is a homeomorphism by Proposition 2.10. Hence, there is a unique homeomorphism $\nu_{H, G}$ making the diagram commute, and one readily checks that the formula is correct. \square

We have now completed the necessary preparations to define the map that was promised in the beginning of this subsection.

Definition 2.21. The map $k_{H, G}$ is the composition

$$\text{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \longrightarrow \tilde{E}(H, G) \times_G * \cong \tilde{E}(H, G)/G \xrightarrow[\nu_{H, G}]{\cong} \mathbf{O}_{\text{gl}}(H, G).$$

Proposition 2.22. *The map $k_{H,G}$ is a weak equivalence.*

Proof. We claim that the G -action on $\tilde{E}(H, G)$ is free. Indeed, if there were α, g, x such that $(\alpha, x)g = (\alpha, x)$, then $xg = x$ would imply that $x \in E(H, G)^{\langle g \rangle}$ where $\langle g \rangle \subseteq G$ denotes the cyclic group generated by g . If $g \neq 1$, then $1 \times \langle g \rangle$ cannot be a graph subgroup of $H \times G$. As $E(H, G)$ is a universal space for the family of graph subgroups, $x \in E(H, G)^{\langle g \rangle} = \emptyset$, a contradiction. Hence, $g = 1$ and the action is free as desired.

The freeness of the action, Theorem A.8, and Lemma A.18 imply that the comparison map between the homotopy quotient and the point-set quotient is a weak equivalence. Thus, $k_{H,G}$ is a weak equivalence. \square

For the reader's convenience, we restate the main result of this section: There is a zig-zag of weak equivalences

$$\mathrm{Orb}(H, G) \xleftarrow[\simeq]{l_{H,G}} \mathrm{Orb}'(H, G) \xrightarrow[\simeq]{k_{H,G}} \mathbf{O}_{\mathrm{gl}}(H, G).$$

The map $l_{H,G}$ and its properties have been established in Subsection 2.3 while $k_{H,G}$ has been discussed in this subsection.

3. The Zig-Zag of Dwyer-Kan Equivalences

In this section, we will upgrade our results on mapping spaces to Dwyer-Kan equivalences. In particular, we will interpret the spaces $\mathrm{Orb}(H, G)$, $\mathrm{Orb}'(H, G)$, and $\mathbf{O}_{\mathrm{gl}}(H, G)$ as mapping spaces of topologically enriched categories and verify that the comparison maps from the previous section give rise to functors sitting in a zig-zag

$$\mathrm{Orb} \xleftarrow[\simeq]{l} \mathrm{Orb}' \xrightarrow[\simeq]{k} \mathbf{O}_{\mathrm{gl}}$$

of Dwyer-Kan equivalences.

Definition 3.1. The topological categories Orb , Orb' , and \mathbf{O}_{gl} all have the set of objects

$$\{G \mid G \text{ a completely universal Lie group}\}$$

and mapping spaces as defined (or suggested by notation) in the previous section.

It remains to specify composition laws and to verify that identity morphisms exist. We will start by reinterpreting the mapping spaces of Orb' and of Orb as realizations of topological groupoids. This allows for a sleek definition of the respective compositions, and we will also be able to verify compatibility with the maps $l_{H,G}$ easily.

Afterward, we will recall the composition law on \mathbf{O}_{gl} and check that it is compatible with the maps $k_{H,G}$. This will imply that our comparison zig-zag from the previous section gives rise to a zig-zag of topologically enriched functors. Recall the following

Definition 3.2. Let \mathcal{C} be topologically enriched category. Then the ordinary category $\pi_0\mathcal{C}$ has $\mathrm{ob} \pi_0\mathcal{C} = \mathrm{ob} \mathcal{C}$ and $(\pi_0\mathcal{C})(c, c') = \pi_0(\mathcal{C}(c, c'))$ with composition defined in the obvious way.

A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ of topologically enriched categories is a *Dwyer-Kan equivalence* if the induced functor $\pi_0 f : \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ is an equivalence of categories and f is a weak equivalence on all mapping spaces.

As we have already verified that the maps in our desired zig-zag of functors are weak equivalences on mapping spaces, and since the object functions are identities, we will immediately be able to deduce that the zig-zag of functors is a zig-zag of Dwyer-Kan equivalences.

Convention. For the remainder of this section, L, K, H, G shall be completely universal Lie groups unless otherwise specified.

3.1. Compositions on Orb' and on Orb

Recall that a *topological groupoid* \mathcal{G} consists of topological spaces $\mathcal{G}_0, \mathcal{G}_1$ together with source and target maps $s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ and a composition $\circ : \mathcal{G}_1 \times_{s,t} \mathcal{G}_1 \rightarrow \mathcal{G}_1$, subject to associativity and identity conditions. The category of topological groupoids is denoted by $\mathrm{Grpd}_{\mathrm{Top}}$.

Definition 3.3. Let G be a topological group and X be a right G -space. Then the *action groupoid* $X//G$ is given by $(X//G)_0 = X$, $(X//G)_1 = X \times G$ with $s(x, g) = x$, $t(x, g) = x \cdot g$ and $(x', g') \circ (x, g) = (x, gg')$.

Given a G -equivariant map $f : K \rightarrow L$ of right G -spaces, the functor $f//G : K//G \rightarrow L//G$ of topological groupoids is defined as $(f//G)_0 = f$ and $(f//G)_1 = f \times G$.

Some straightforward computations prove

Proposition 3.4. *The construction from before gives rise to a functor $-//G : G - \text{Top} \rightarrow \text{Grpd}_{\text{Top}}$. Furthermore, the following diagram commutes*

$$\begin{array}{ccccc} G - \text{Top} & \xrightarrow{-//G} & \text{Grpd}_{\text{Top}} & \xrightarrow{N_\bullet} & \text{Top}^\Delta & \xrightarrow{|\cdot|} & \text{Top} \\ & & & & \searrow & \nearrow & \\ & & & & - \times_G EG & & \end{array}$$

where N_\bullet is the topologically enriched version of the nerve functor.

Thus, we see that the morphism spaces of Orb' and of Orb can be described as realizations of certain action groupoids, namely

$$\text{Orb}(H, G) = |N_\bullet(\text{map}(H, G)//G)|, \quad \text{Orb}'(H, G) = |N_\bullet(\tilde{E}(H, G)//G)|.$$

We proceed by defining composition laws on the level of action groupoids.

Definition 3.5. (i) The functor $\diamond : (\text{map}(H, G)//G) \times (\text{map}(K, H)//H) \rightarrow \text{map}(K, G)//G$ is defined by composition in level 0 and by the formula

$$\begin{array}{ccc} \text{map}(H, G) \times G \times \text{map}(K, H) \times H & \longrightarrow & \text{map}(K, G) \times G \\ (\beta, g), (\alpha, h) & \mapsto & (\beta \circ \alpha, \beta(h)g) \end{array}$$

in level 1.

(ii) The functor $\diamond : (\tilde{E}(H, G)//G) \times (\tilde{E}(K, H)//H) \rightarrow \tilde{E}(K, G)//G$ is defined by

$$(\beta, y), (\alpha, x) \mapsto (\beta \circ \alpha, x \circ y)$$

in level 0 and by

$$(\beta, y, g), (\alpha, x, h) \mapsto (\beta \circ \alpha, x \circ y, \beta(h)g)$$

in level 1 where $\beta \in \text{map}(H, G)$, $y \in E(H, G)^\beta$, $g \in G$, $\alpha \in \text{map}(K, H)$, $x \in E(K, H)^\alpha$, $h \in H$. In both formulae, we use the composition of \mathcal{L} which is the underlying space of both $E(H, G)$ and $E(K, H)$.

Remark 3.6. Denote by $\mathbb{B}G = */G$ the action groupoid associated to the unique action of G on the one-point space $*$. Then one can show that $\text{map}(H, G)//G$ is isomorphic to the internal mapping topological groupoid $\text{map}_{\text{Grpd}_{\text{Top}}}(\mathbb{B}H, \mathbb{B}G)$, and the composition \diamond from part (i) of the previous definition is precisely the composition coming from this enrichment of Grpd_{Top} over itself.

Proposition 3.7. *The maps from the previous definition are functors.*

Proof. We have to verify well-definedness and compatibility with composition and identities for both candidate functors \diamond . We will do this for the second one only because the necessary verifications for the first one are part of the verifications for the second one.

In order to check well-definedness in level 0, we have to prove that for β, y, α, x as above, $x \circ y \in E^{\beta \circ \alpha}$. This means that $x \circ y$ is fixed under the action of $\text{graph}(\beta \circ \alpha) \subseteq K \times G$. Pick $k \in K$. We would like to show that $(x \circ y)(k, \beta(\alpha(k))) = (x \circ y)$. Chose $v \in \mathbb{R}^\infty$. Spelling out the left hand side and evaluating at v , we obtain

$$(x \circ y)(k, \beta(\alpha(k)))(v) = k^{-1} \cdot (x \circ y)(\beta(\alpha(k))v) = k^{-1} \cdot (x(y(\beta(\alpha(k))v))).$$

We have $y \in E^\beta$ by assumption and $\alpha(k) \in H$. Thus, $(\alpha(k), \beta(\alpha(k))) \in \text{graph}(\beta)$ and $y \cdot (\alpha(k), \beta(\alpha(k))) = y$. Concretely,

$$\alpha(k)^{-1} \cdot y(\beta(\alpha(k))v) = y(v)$$

and, therefore, $y(\beta(\alpha(k))v) = \alpha(k)y(v)$. Plugging this into the previous computation,

$$(x \circ y)(k, \beta(\alpha(k)))(v) = k^{-1}(x(\alpha(k)y(v))).$$

As $x \in E^\alpha$, we have $x \cdot (k, \alpha(k)) = x$. Evaluating at $y(v) \in \mathbb{R}^\infty$ proves that the right hand side becomes $x(y(v))$, and we have proven that $x \circ y \in E^{\beta \circ \alpha}$.

For the well-definedness in level 1, it is left to verify that the morphism of $\tilde{E}(K, G)/G$ specified by $(\beta \circ \alpha, x \circ y, \beta(h)g)$ has the correct source and the correct target. The source is $(\beta \circ \alpha, x \circ y) = s(\beta, y, g) \diamond s(\alpha, x, h)$ as desired. The target is $((\beta \circ \alpha) \cdot (\beta(h)g), (x \circ y)(\beta(h)g))$ and this should agree with $t(\beta, y, g) \diamond t(\alpha, x, h) = (\beta \cdot g, y \cdot g) \diamond (\alpha \cdot h, x \cdot h) = ((\beta \cdot g) \circ (\alpha \cdot h), (x \cdot h) \circ (y \cdot g))$. One readily calculates that $(\beta \circ \alpha) \cdot (\beta(h)g) = (\beta \cdot g) \circ (\alpha \cdot h) \in \text{map}(K, G)$, so the first components agree. For the second components, pick $v \in \mathbb{R}^\infty$. We have

$$((x \cdot h) \circ (y \cdot g))(v) = (x \cdot h)(y(gv)) = x(hy(gv)).$$

As before, $y \in E^\beta$, and $hy(gv) = y(\beta(h)gv)$ consequently. Plugging this in,

$$((x \cdot h) \circ (y \cdot g))(v) = x(y(\beta(h)gv)) = (x \circ y)(\beta(h)gv) = ((x \circ y)(\beta(h)g))(v).$$

Thus, the candidate functor \diamond is compatible with sources and targets.

Given an object $((\beta, y), (\alpha, x))$ in the product groupoid on the left hand side, its identity is the morphism $((\beta, y, 1), (\alpha, x, 1))$, and applying \diamond yields $(\beta \circ \alpha, x \circ y, 1)$ which is the identity of the object $(\beta \circ \alpha, x \circ y)$.

Pick two composable morphisms $((\beta, y, g), (\alpha, x, h))$ and $((\beta \cdot g, y \cdot g, g'), (\alpha \cdot h, x \cdot h, h'))$. Their composition inside the groupoid on the left hand side is $((\beta, y, gg'), (\alpha, x, hh'))$ which is mapped to $(\beta \circ \alpha, x \circ y, \beta(hh')gg')$ by \diamond . On the other hand, the composition inside the groupoid on the right hand side of the individual images under \diamond is

$$\begin{aligned} & ((\beta \cdot g) \circ (\alpha \cdot h), (x \cdot h) \circ (y \cdot g), ((\beta \cdot g)(h'))g') \circ (\beta \circ \alpha, x \circ y, \beta(h)g) \\ &= (\beta \circ \alpha, x \circ y, \beta(h)g((\beta \cdot g)(h'))g') \\ &= (\beta \circ \alpha, x \circ y, \beta(h)gg^{-1}\beta(h')gg') \end{aligned}$$

which agrees with $(\beta \circ \alpha, x \circ y, \beta(hh')gg')$. \square

Proposition 3.8. *The composition laws \diamond defined before give rise to categories Orb_{Grpd} and $\text{Orb}'_{\text{Grpd}}$ enriched in topological groupoids with*

$$\text{ob } \text{Orb}_{\text{Grpd}} = \text{ob } \text{Orb}'_{\text{Grpd}} = \text{ob } \text{Orb} = \text{ob } \text{Orb}'$$

and

$$\text{Orb}_{\text{Grpd}}(H, G) = \text{map}(H, G)/G, \quad \text{Orb}'_{\text{Grpd}}(H, G) = \tilde{E}(H, G)/G.$$

Furthermore, the maps $\tilde{p}_{H,G} : \tilde{E}(H, G) \longrightarrow \text{map}(H, G)$ from Definition 2.12 give rise to a functor $p : \text{Orb}'_{\text{Grpd}} \longrightarrow \text{Orb}_{\text{Grpd}}$ which is the identity on objects and $p_{H,G} := \tilde{p}_{H,G}/G$ on morphism groupoids.

Proof. We have to verify that the composition laws \diamond are associative and unital. Again, we will do this for the second case $\text{Orb}'_{\text{Grpd}}$ only.

Unitality means that there is a, necessarily unique, functor $* \longrightarrow \text{Orb}'_{\text{Grpd}}(G, G)$ for the topological groupoid $*$ consisting of one object $*$ and its identity. Specifying such a functor amounts to the selection of an object

$$\text{id}_G \in \text{Orb}'_{\text{Grpd}}(G, G)_0 = \tilde{E}(G, G) = \{(\alpha, x) \in \text{map}(G, G) \times E(G, G) \mid x \in E^\alpha\}.$$

The underlying space of $E(G, G)$ is just $\mathcal{L} = \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ and it is easily verified that $\text{id} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \in \mathcal{L}$ lives in the subspace E^{id} . Thus, we claim that $\text{id}_G := (\text{id}, \text{id})$ yields a unit for \diamond .

Let $(\alpha, x) \in \text{Orb}'_{\text{Grpd}}(H, G)$. Then $\text{id}_G \diamond (\alpha, x) = (\text{id}, \text{id}) \diamond (\alpha, x) = (\text{id} \circ \alpha, x \circ \text{id}) = (\alpha, x)$ as desired. Similarly, $(\alpha, x) \diamond \text{id}_G = (\alpha, x)$. On the level of morphisms, the functor $* \longrightarrow \text{Orb}'_{\text{Grpd}}(G, G)$ selects the identity of id_G which is the morphism $(\text{id}, \text{id}, 1) \in \text{Orb}'_{\text{Grpd}}(G, G)(\text{id}_G, \text{id}_G) \subseteq \tilde{E}(G, G) \times G$. For a morphism $(\alpha, x, h) \in \text{Orb}'_{\text{Grpd}}(H, G)$, we compute $(\text{id}, \text{id}, 1) \diamond (\alpha, x, h) = (\text{id} \circ \alpha, x \circ \text{id}, \text{id}(h)1) = (\alpha, x, h)$ as desired. For right-unitality, we have $(\alpha, x, h) \diamond (\text{id}, \text{id}, 1) = (\alpha, x, \alpha(1)h) = (\alpha, x, h)$ using the identity of id_H .

We prove associativity on the level of morphisms only because the statement on objects follows from that. Thus, pick $(\alpha, x, k) \in \text{Orb}'_{\text{Grpd}}(L, K)_1$, $(\beta, y, h) \in \text{Orb}'_{\text{Grpd}}(K, H)_1$, $(\gamma, z, g) \in \text{Orb}'_{\text{Grpd}}(H, G)_1$. We calculate

$$\begin{aligned}
& ((\gamma, z, g) \diamond (\beta, y, h)) \diamond (\alpha, x, k) \\
&= (\gamma \circ \beta, y \circ z, \gamma(h)g) \diamond (\alpha, x, k) \\
&= (\gamma \circ \beta \circ \alpha, x \circ y \circ z, (\gamma \circ \beta)(k)\gamma(h)g) \\
&= (\gamma \circ \beta \circ \alpha, x \circ y \circ z, \gamma(\beta(k)h)g) \\
&= (\gamma, z, g) \diamond (\beta \circ \alpha, x \circ y, \beta(k)h) \\
&= (\gamma, z, g) \diamond ((\beta, y, h) \diamond (\alpha, x, k))
\end{aligned}$$

concluding the verification of associativity.

Spelling out the definitions, we see that the map $p_{H,G} = \tilde{p}_{H,G}/G$ of topological groupoids is given by

$$\begin{array}{ccc}
\text{Orb}'_{\text{Grpd}}(H, G)_0 & \longrightarrow & \text{Orb}_{\text{Grpd}}(H, G)_0 \\
(\alpha, x) & \mapsto & \alpha
\end{array}, \quad
\begin{array}{ccc}
\text{Orb}'_{\text{Grpd}}(H, G)_1 & \longrightarrow & \text{Orb}_{\text{Grpd}}(H, G)_1 \\
(\alpha, x, g) & \mapsto & (\alpha, g)
\end{array}$$

and it is evident that these formulae are compatible with the composition laws in Orb_{Grpd} and $\text{Orb}'_{\text{Grpd}}$, respectively, as H, G vary. Therefore, the maps $p_{H,G}$ assemble into a functor $p : \text{Orb}'_{\text{Grpd}} \longrightarrow \text{Orb}_{\text{Grpd}}$ as claimed. \square

Proposition 3.9. *Applying the functor $|\cdot| \circ N_\bullet$ as in Proposition 3.4 to the composition laws from Definition 3.5 gives rise to categories Orb' and Orb enriched in topological spaces.*

Moreover, the functor p from Proposition 3.8 yields a topologically enriched functor $l : \text{Orb}' \longrightarrow \text{Orb}$ whose mapping space components $l_{H,G} : \text{Orb}'(H, G) \longrightarrow \text{Orb}(H, G)$ are exactly the maps $l_{H,G}$ from Definition 2.17. In particular, the functor l is a Dwyer-Kan equivalence.

Proof. Both $|\cdot|$ and N_\bullet are strongly monoidal. This is straightforward for N_\bullet . A reference for the monoidality of $|\cdot|$ is [9, Corollary 11.6].

Therefore, applying these functors to the individual mapping spaces of $\text{Orb}'_{\text{Grpd}}$ and Orb_{Grpd} and keeping the object sets the same turns these categories enriched over topological groupoids into categories Orb' and Orb enriched over topological spaces.

We have $|N_\bullet \text{Orb}'(H, G)| = |N_\bullet \tilde{E}(H, G)| = \tilde{E}(H, G) \times_G EG$ and $|N_\bullet \text{Orb}(H, G)| = \text{map}(H, G) \times_G EG$ by Proposition 3.4. Hence, the mapping spaces $\text{Orb}'(H, G)$ and $\text{Orb}(H, G)$ agree with our definitions from Section 2.

Applying $|\cdot| \circ N_\bullet$ to p yields a functor $l : \text{Orb}' \longrightarrow \text{Orb}$. As the mapping groupoid components $p_{H,G} : \text{Orb}'_{\text{Grpd}}(H, G) \longrightarrow \text{Orb}_{\text{Grpd}}(H, G)$ of p come from maps $\tilde{p}_{H,G}$ of G -spaces, we may apply Proposition 3.4 again and deduce that the mapping space components of $l = |N_\bullet(p)|$ are $|N_\bullet(p)|_{H,G} = |N_\bullet(p_{H,G})| = \tilde{p}_{H,G} \times_G EG$ which agree with the maps $l_{H,G}$ from Definition 2.17. These are weak equivalences by Proposition 2.18. Consequently, l is a Dwyer-Kan equivalence. \square

Remark 3.10. *In [4, Remark 4.3], Gepner and Henriques choose $\text{Orb}(H, G)$ to be the fat geometric realization $||N_\bullet(\text{map}(H, G)/G)||$. Let us denote this space by $\text{Orb}_{||\cdot||}(H, G)$ for the moment. For any good simplicial space X , the canonical map $||X|| \longrightarrow |X|$ is a homotopy equivalence, see [13, Proposition A.1.(iv)]. In particular, $\text{Orb}_{||\cdot||}(H, G) \longrightarrow \text{Orb}(H, G)$ is a homotopy equivalence.*

The composition in $\text{Orb}_{||\cdot||}$ depends on a choice of maps $i : ||X \times Y|| \longrightarrow ||X|| \times ||Y||$ and $r : ||X|| \times ||Y|| \longrightarrow ||X \times Y||$ such that $||X \times Y||$ is a retract of $||X|| \times ||Y||$ via these maps and such that certain associativity conditions hold, see [4, Remarks 2.23 and 4.3]. If one chooses these maps in such a way that the diagram

$$\begin{array}{ccc}
||X|| \times ||Y|| & \xrightarrow{\tau} & ||X \times Y|| \\
\downarrow \simeq & & \downarrow \simeq \\
|X| \times |Y| & \xrightarrow{\cong} & |X \times Y|
\end{array}$$

commutes, then one obtains a Dwyer-Kan equivalence $\text{Orb}_{||\cdot||} \longrightarrow \text{Orb}$.

3.2. Compatibility with the Composition on \mathbf{O}_{gl}

We begin by recalling the composition on \mathbf{O}_{gl} . Having established explicit descriptions of the maps and objects in question, it will be straightforward to verify that the remaining maps $k_{H,G}$ are compatible with identities and composition.

Definition 3.11 ([12, Construction 2.25]). The topologically enriched category \mathbf{O}_{gl} has objects as in Definition 3.1 and mapping spaces $\mathbf{O}_{\text{gl}}(H, G) = (E(H, G)/G)^H$ as defined in Definition 2.6. Composition is defined as

$$\begin{array}{ccc} \mathbf{O}_{\text{gl}}(H, G) \times \mathbf{O}_{\text{gl}}(K, H) & \longrightarrow & \mathbf{O}_{\text{gl}}(K, G) \\ ([y], [x]) & \mapsto & [x \circ y]. \end{array}$$

The identity morphism in $\mathbf{O}_{\text{gl}}(G, G)$ is given by $[\text{id}_{\mathbb{R}^\infty}]$, the class of the neutral element of the topological monoid $\mathcal{L} = \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$, which is the underlying space of $\tilde{E}(G, G)$.

Proposition 3.12. *The maps $k_{H,G} : \text{Orb}'(H, G) \longrightarrow \mathbf{O}_{\text{gl}}(H, G)$ from Definition 2.21 gives rise to a functor $k : \text{Orb}' \longrightarrow \mathbf{O}_{\text{gl}}$ that is the identity on objects.*

Proof. Recall that $k_{H,G}$ is defined as the composition

$$\text{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \longrightarrow \tilde{E}(H, G) \times_G * \cong \tilde{E}(H, G)/G \xrightarrow[\nu_{H,G}]{\cong} \mathbf{O}_{\text{gl}}(H, G).$$

where $\nu_{H,G}$ is given by $[\alpha, x] \mapsto [x]$, see Proposition 2.20, and the first map is given by $[\alpha, x, u] \mapsto [\alpha, x]$.

We have used the bar construction model $|N_\bullet(\tilde{E}(H, G)/G)|$ for $\tilde{E}(H, G) \times_G EG = \text{Orb}'(H, G)$, and an arbitrary element of this space is of the form $[\alpha, x, \vec{g}, s]$ for $(\alpha, x) \in \tilde{E}(H, G)$, $\vec{g} \in G^n$, $s \in \Delta^n$. The map $\tilde{E}(H, G) \times_G EG \longrightarrow \tilde{E}(H, G)/G$ becomes $[\alpha, x, \vec{g}, s] \mapsto [\alpha, x]$ under this implicit isomorphism. We conclude that the composite map $k_{H,G}$ has the formula $[\alpha, x, \vec{g}, s] \mapsto [x]$.

In this description, the composition on Orb' , which is given by realizing the maps \diamond from Definition 3.5, is given by

$$\begin{array}{ccc} \text{Orb}'(H, G) \times \text{Orb}'(K, H) & \longrightarrow & \text{Orb}'(K, G) \\ [\beta, y, \vec{g}, t], [\alpha, x, \vec{h}, s] & \mapsto & [\beta \circ \alpha, x \circ y, ?] \end{array}$$

where the unspecified value $?$ is irrelevant to our following considerations.

Namely, we need to verify that k is compatible with identities and composition. We leave the straightforward computation for identities to the inclined reader and proceed to the composition. Consider the following diagram:

$$\begin{array}{ccc} \text{Orb}'(H, G) \times \text{Orb}'(K, H) & \xrightarrow{\circ_{K,H,G}} & \text{Orb}'(K, G) \\ \downarrow k_{H,G} \times k_{K,H} & & \downarrow k_{K,G} \\ \mathbf{O}_{\text{gl}}(H, G) \times \mathbf{O}_{\text{gl}}(K, H) & \xrightarrow{\circ_{K,H,G}} & \mathbf{O}_{\text{gl}}(K, G) \end{array}$$

Starting with $[\beta, y, \vec{g}, t], [\alpha, x, \vec{h}, s]$ in the upper left corner, we check that it is mapped to $[x \circ y]$ under $k_{K,G} \circ (\circ_{K,H,G})$ which is equal to its value under $\circ_{K,H,G} \circ (k_{H,G} \times k_{K,H})$. This concludes the proof. \square

Corollary 3.13. *There is a zig-zag of Dwyer-Kan equivalences*

$$\text{Orb} \xleftarrow[l]{\simeq} \text{Orb}' \xrightarrow[k]{\simeq} \mathbf{O}_{\text{gl}}$$

as promised at the beginning of this section.

Proof. The functor l exists and is a Dwyer-Kan equivalence by Proposition 3.9. The existence of k was discussed in the previous Proposition. It is a Dwyer-Kan equivalence by Proposition 2.22. \square

A. Appendix

This appendix is concerned with several technical topics which be deemed to be too distracting for the flow of arguments to be included in the main part of this exposition.

After stating our conventions on the usage of the adjectives *compact* and *compactly generated* in Subsection A.1, we proceed by providing helpful statements on the interaction of closed inclusions with colimits in CGWH spaces in Subsection A.2. Afterwards, we will recall *normal spaces* and show that EG is normal in preparation for the following subsection.

Subsection A.4 will show that for a free G -space, which is Hausdorff, the comparison map from the homotopy quotient to the point-set quotient is a weak equivalence (Theorem A.8). This result is well-known for manifolds or free G -CW-complexes, but we could not find a sufficiently general statement in the literature that applies to our circumstances. Therefore, we provide a proof that works for every compactly generated Hausdorff space.

In the following Subsection A.5, we will investigate the topology of the spaces $\tilde{E}(H, G)$ in order to finalize some proofs from the main body of this paper.

In contrast to these point-set issues, the remaining two subsections are of a model categorical nature. They are devoted to proving that Dwyer-Kan equivalences of topologically enriched categories induce Quillen equivalences of associated (topologically enriched) presheaf categories. This will be shown in the last subsection (A.7) after introducing a preparatory result in Subsection A.6.

A.1. Compactly Generated Weak Hausdorff Spaces

The main body of this paper takes place in the category of compactly generated weak Hausdorff spaces, also referred to CGWH spaces. Before we deal with the necessary point-set arguments, let us make the used terminology precise.

A space is *compact* if every open cover admits a finite subcover. This is also being referred to as *quasi-compact* in other sources which include the Hausdorff property into the definition of compactness.

Moreover, a space X is *compactly generated* if, for any subset $Y \subseteq X$, Y is closed if and only if $u^{-1}(Y)$ is closed for every compact Hausdorff K and every continuous $u : K \rightarrow X$. The space X is weak Hausdorff if for every such u and K , the image $u(K)$ is closed in X .

These definitions are taken from [15]. Note that this terminology varies within the literature, and some sources refer to compactly generated spaces as *k-spaces* while they take compactly generated spaces to be compactly generated weak Hausdorff spaces in our sense.

Note that the property of being compactly generated is a local property, i.e., a space is compactly generated if and only if each point has a compactly generated neighborhood. The property of being weak Hausdorff is not local, though.

In this paper, we refer to CGWH spaces as (*topological*) *spaces* and denote the corresponding category by \mathbf{Top} . Within the next subsections, we will have to deal with their point-set subtleties and cite statements about not necessarily CGWH spaces. To this end, we will use the term *general topological space* for a space that is not necessarily CGWH.

The category of CGWH spaces is cocomplete. Limits and colimits may, however, differ from those computed in the category of general topological spaces. Our convention is that limits and colimits are computed in CGWH unless it is explicitly declared that the diagram in consideration lives in the category of general topological spaces. In this case, limits and colimits are to be taken in the category of general topological spaces. The latter situation does only occur in Subsections A.2–A.5.

For the special case of products, we adopt the following notation from [15]: Given two spaces X and Y , we denote by $X \times_0 Y$ the product taken in the category of general topological spaces, which is not necessarily compactly generated. In contrast, $X \times Y$ shall denote the product in the category of CGWH spaces.

A.2. Closed Inclusions and CGWH Colimits

We will now shed some light on situations where specific colimits agree regardless of whether they are computed in CGWH or in the category of general topological spaces.

Lemma A.1 ([7, Section 2.4, p. 59]). *The category of topological spaces is cocomplete. In the case of pushouts along closed inclusions or transfinite compositions of injections, colimits may be computed in the category of general topological spaces since they are already weak Hausdorff.*

Lemma A.2. *In the category of topological spaces, closed inclusions are closed under pushouts, transfinite compositions, and retracts.*

Proof. As weak Hausdorff spaces are automatically T_1 , a closed inclusion in the category of topological spaces is a closed T_1 inclusion in the category of general topological spaces. Retracts of maps of topological spaces are also retracts of maps of general topological spaces. Also, the relevant pushouts and transfinite compositions can be computed in the category of general topological spaces.

The claim follows from the proof of [7, Lemma 2.4.5] for the cases of pushouts and transfinite compositions and from the proof of [7, Corollary 2.4.6] for the case of retracts. \square

Lemma A.3. *Let λ be a limit ordinal and $X : \lambda \longrightarrow \text{Top}$ be a λ -sequence along closed inclusions. Furthermore, let K be a compact space. Then any map $K \longrightarrow \text{colim}_{\beta < \lambda} X_\beta$ factors through some X_μ , $\mu < \lambda$. In particular, the canonical map*

$$\text{colim}_{\beta < \lambda} \pi_k(X_\beta) \longrightarrow \pi_k(\text{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism.

Proof. The colimit can be computed in the category of general topological spaces. As in the proof of the previous lemma, a closed inclusion is a closed T_1 inclusion of general topological spaces. Also, note that the cofinality of λ is infinite because λ is a limit ordinal. In particular, λ is γ -filtered for each finite cardinal γ in the sense of [7, Definition 2.1.2]. Therefore, [7, Proposition 2.4.2] tells us that the canonical map

$$\text{colim}_{\beta < \lambda} \text{Top}(K, X_\beta) \longrightarrow \text{Top}(K, \text{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism, proving the first claim. The second claim follows by a standard argument. \square

A.3. Normal Spaces

In this subsection, we will recall the definition of a normal space. Afterward, we will show that normal spaces are abundant and that the class of normal spaces, in contrast to the class of Hausdorff spaces, is closed under certain colimit constructions. As normality implies Hausdorffness for T_1 spaces, we will use our results to deduce that EG and $\tilde{E}(H, G)$ are Hausdorff spaces (Lemma A.7 and Lemma A.18 respectively).

Definition A.4. A general topological space X is *normal* if for any two disjoint closed subsets $A, B \subseteq X$, there are open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$, $A \subseteq U, B \subseteq V$.

There is a very useful characterization of normal spaces:

Theorem A.5 (Tietze Extension Theorem). *X is normal if and only if any continuous map $A \longrightarrow \mathbb{R}$, $A \subseteq X$ closed, can be extended to a map $X \longrightarrow \mathbb{R}$.*

This theorem easily allows us to deduce a few properties of and recognition criteria for normal spaces:

Lemma A.6. (i) *Metrizable spaces and compact Hausdorff spaces are normal.*

(ii) *Closed subspaces of normal spaces are normal.*

(iii) *If $A \subseteq B$ is a closed inclusion, B and X are normal spaces, then for any pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array} \quad \Gamma.$$

the space Y is normal.

(iv) *If λ is an ordinal and $(X_\beta)_{\beta < \lambda}$ is a λ -sequence of normal spaces along closed inclusions, then the space $\text{colim}_{\beta < \lambda} X_\beta$ is normal.*

(v) *Normal T_1 spaces are Hausdorff.*

(vi) Retracts of normal T_1 spaces are normal.

(vii) Quillen cofibrant topological spaces are normal.

Proof. (i) is classical. (ii)-(iv) can easily be shown using the Tietze Extension Theorem.

For example, let us show (iv): Given $(X_\beta)_{\beta < \lambda}$, a closed subspace $A \subseteq \operatorname{colim}_{\beta < \lambda} X_\beta =: X$, and a continuous map $f : A \rightarrow \mathbb{R}$, we wish to find an extension $F : X \rightarrow \mathbb{R}$ of f . We start a transfinite recursion in degree 0. As $A \cap X_0$ is closed and X_0 is normal, we may find an extension $F_0 : X_0 \rightarrow \mathbb{R}$ of $f|_{A \cap X_0}$. Assume that $\beta = \gamma + 1$ is a successor ordinal and that there is an extension $F_\gamma : X_\gamma \rightarrow \mathbb{R}$ of $f|_{A \cap X_\gamma}$. Then $X_\gamma \cup (A \cap X_\beta) \subseteq X_\beta$ is closed (because the map $X_\gamma \rightarrow X_\beta$ has closed image), and the function $F_\gamma \cup f|_{A \cap X_\beta} : X_\gamma \cup (A \cap X_\beta) \rightarrow \mathbb{R}$ is continuous (because $X_\gamma \rightarrow X_\beta$ is a homeomorphism onto its image). Hence, by the normality of X_β , we may extend this function to $F_\beta : X_\beta \rightarrow \mathbb{R}$. If β is a limit ordinal and we have already chosen compatible extensions $F_\gamma : X_\gamma \rightarrow \mathbb{R}$, then $X_\beta = \operatorname{colim}_{\gamma < \beta} X_\gamma$ and we extend $f|_{A \cap X_\beta}$ by $\operatorname{colim}_{\gamma < \beta} F_\gamma : X_\beta \rightarrow \mathbb{R}$. Finally, taking $F := \operatorname{colim}_{\beta < \lambda} F_\beta : X \rightarrow \mathbb{R}$ yields the desired extension of f .

Part (v) is immediate because points are closed in T_1 spaces. For part (vi), if $A \xrightarrow{i} X \xrightarrow{p} A$ is a retract diagram, then i is automatically a homeomorphism onto its image. As X is Hausdorff, its retract subspace $i(A)$ is closed. Therefore, it is normal by (ii). The last part follows from (i), (iii), (iv), (vi), and Lemma A.19. \square

Lemma A.7. *Let G be a compact Lie group. Then EG is normal and Hausdorff.*

Proof. The space EG is the geometric realization of the simplicial topological space EG_\bullet which is given by $EG_n = G^{n+1}$. As for every geometric realization, EG is the sequential colimit of its skeleta $\operatorname{sk}_n EG$, which sit in pushouts

$$\begin{array}{ccc} L_n EG_\bullet \times \Delta^n \cup_{L_n EG_\bullet \times \partial \Delta^n} EG_n \times \partial \Delta^n & \longrightarrow & \operatorname{sk}_{n-1} EG \\ \downarrow & \lrcorner & \downarrow \\ EG_n \times \Delta^n & \longrightarrow & \operatorname{sk}_n EG \end{array}$$

where $L_n EG_\bullet \rightarrow EG_n$ is the n -th latching map. Since EG_\bullet is proper, cf. the Proof of Proposition 2.18, the latching maps are closed Hurewicz cofibration. The same is true for the vertical pushout-product morphism on the left hand side. In particular, these pushout-product morphisms are closed inclusions. As $EG_n \times \Delta^n = G^{n+1} \times \Delta^n$ and $\operatorname{sk}_{-1} EG_\bullet = \emptyset$ are normal, it follows inductively from Lemma A.6 that the $\operatorname{sk}_n EG$ are normal.

Moreover, the pushout diagram tells us that the morphisms $\operatorname{sk}_{n-1} EG \rightarrow \operatorname{sk}_n EG$ are closed inclusions, using Lemma A.2. Hence, $EG = \operatorname{colim}_n \operatorname{sk}_n EG$ is normal by Lemma A.6. As EG is CGWH, it is T_1 . Thus, by Lemma A.6, it is Hausdorff. \square

A.4. Homotopy Quotients of Free G -Spaces

The goal of this subsection is proving the following theorem:

Theorem A.8. *Let G be a compact Lie group and X a compactly generated Hausdorff space, endowed with a free right G -action. Then the canonical map $X \times_G EG \rightarrow X/G$ from the homotopy quotient to the quotient is a weak equivalence.*

The crucial point in the proof of this theorem is gaining homotopical control over the quotient maps $X \times EG \rightarrow X \times_G EG$ and $X \rightarrow X/G$. Let us begin by citing several facts about G -spaces.

Theorem A.9. *Let G be a compact Lie group acting on a general topological space X . Denote by $q : X \rightarrow X/G$ the quotient map. Then the following hold:*

- (i) *For any closed subset $A \subset X$, the set $AG = \{ag \mid a \in A, g \in G\}$ is closed in X .*
- (ii) *The quotient map q is closed.*
- (iii) *Let A be any G -invariant subset of X . Then the birestriction $A \rightarrow q(A)$ of q agrees with the quotient map $A \rightarrow A/G$, i.e., the subspace topology on $q(A)$ is the quotient topology of A/G .*
- (iv) *If X is a CGWH space, then so is X/G . Put differently, the quotient computed in general topological spaces agrees with the quotient taken in CGWH spaces.*

(v) If X is Hausdorff, then the quotient map q is proper.

(vi) If X is Hausdorff, then so is X/G .

(vii) Assume that the following two conditions are satisfied:

a) For some fixed $H \leq G$, the stabilizer subgroups of all points in X are conjugate to H .

b) The space X is Tychonoff, i.e., completely regular and Hausdorff.

Then q is a fiber bundle with typical fiber $H \backslash G$.

Proof. Statement (i) is a special case of [16, Proposition I.3.1.(iii)], and the second assertion (ii) follows from (i) by the definition of the quotient topology. An alternative argument can be found in [11, Proposition A.1.20.(i)]. The next two statements (iii)-(iv) are precisely [11, Proposition A.1.20.(ii)-(iii)].

The crucial step in proving (v) is ensuring that the fibers $q^{-1}(q(x))$ are compact. This is rendered possible by the Hausdorff property because it implies that, for any $x \in X$, the canonical map $\text{Stab}(x) \backslash G \rightarrow xG = q^{-1}(q(x))$ is a homeomorphism. A full argument can be found in [16, Section I.3] or [2, Section I.3]. These two sources also provide a reference for (vi). Note that [2] exclusively works in general topological spaces that are Hausdorff.

Finally, condition (a) is a rephrasing of the requirement that all orbits have type $H \backslash G$, therefore [2, Theorem II.5.8] applies verbatim. \square

However, the point-set property of being Tychonoff is very inconvenient from a homotopy theorist's point of view because one does not seem to have any grasp on it under limits. Complete regularity means that closed sets can be separated from points by Urysohn functions. Most limits in CGWH spaces necessitate usage of the k -ification functor, which potentially introduces more closed sets and, therefore, destroys complete regularity.

On the other hand, compact Hausdorff spaces are automatically Tychonoff, and the topology on a CGWH space is generated by its compact Hausdorff subspaces. This will be the main argument in the proof of the following theorem.

Theorem A.10. *Let G be a compact Lie group acting on a compactly generated Hausdorff space X . Assume that for some fixed $H \leq G$, all stabilizers subgroups of points in X are conjugate to H . Then the quotient map $q : X \rightarrow X/G$ is a fibration.*

Proof. The topology on X is generated by its compact subspaces, which are automatically Hausdorff, in the sense that the canonical map

$$\text{colim}_{K \subseteq X_{\text{compact}}} K \rightarrow X$$

is a homeomorphism. For any compact set $K \subseteq X$, there is a G -invariant, compact superset KG . In particular, the G -invariant, compact sets form a cofinal subcategory of the compact sets, and the natural map

$$\text{colim}_{\substack{K \subseteq X_{\text{compact}} \\ \text{and } G\text{-invariant}}} K \rightarrow X$$

is a homeomorphism. This colimit commutes with taking G -quotients. Hence,

$$X/G = \text{colim}_{\substack{K \subseteq X_{\text{compact}} \\ \text{and } G\text{-invariant}}} (K/G).$$

For any G -invariant, compact $K \subseteq X$, the space K is compact Hausdorff which implies normal Hausdorff and, consequently, the Tychonoff property. The stabilizer subgroups of the G -action on K are conjugate to a fixed H , so we may apply Theorem A.9.(vii) and deduce that the quotient map $K \rightarrow K/G$ is a fiber bundle and, therefore, a fibration. Moreover, note that this quotient map is exactly the birestriction $q|_K$ of $q : X \rightarrow X/G$ to K and its image $q(K) = K/G$, see Theorem A.9.(iv).

Having made the necessary preparations, let us consider a generating acyclic cofibration of topological spaces $i : A \hookrightarrow B$ and a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & & \downarrow q \\ B & \xrightarrow{g} & X/G \end{array}$$

As B is compact, its image $g(B) \subseteq X/G$ is compact. Since q is proper by Theorem A.9.(v), the G -invariant subset $K := q^{-1}(g(B))$ is compact, too. Also, $g(B) = q(q^{-1}(g(B))) = q(K) = K/G$. By corestriction, our lifting problem reduces to the left-hand square of

$$\begin{array}{ccccc} A & \xrightarrow{f} & K & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow q|_K & & \downarrow q \\ B & \xrightarrow{g} & K/G & \longrightarrow & X/G \end{array}$$

The middle vertical map $q|_K$ is a fibration by our previous considerations, and we may find a lift as indicated. Postcomposing with the inclusion $K \hookrightarrow X$ immediately solves our original lifting problem. \square

Remark A.11. In the previous proof, the Hausdorff property for X ensures that the space $K = q^{-1}(g(B))$ is compact Hausdorff. Compactness follows from the properness of q , which necessitates the Hausdorff property, and the Hausdorff property for K itself is inherited from X . Despite several efforts, we were unable to eliminate the Hausdorff requirement for X . However, we believe that the statement in its current form may prove useful for other applications as compact Lie group actions on Hausdorff spaces are ubiquitous.

Before we proceed to the proof of Theorem A.8, let us record some preparations which will help us to establish isomorphisms on π_0 and π_1 .

Proposition A.12. Let X and G be as before. The G -action on X induces a $\pi_0(G)$ -action on $\pi_0(X)$, and there is a dashed arrow making the following triangle commute:

$$\begin{array}{ccc} \pi_0(X) & \xrightarrow{\pi_0(q)} & \pi_0(X/G) \\ \downarrow \varrho & \nearrow \overline{\pi_0(q)} & \\ \pi_0(X)/\pi_0(G) & & \end{array}$$

This arrow is an isomorphism and natural with respect to morphisms of G -spaces.

Proof. The $\pi_0(G)$ -action on $\pi_0(X)$ is well-defined because π_0 commutes with products, and G -morphisms $X \rightarrow Y$ give rise to morphisms $\pi_0(X) \rightarrow \pi_0(Y)$ of $\pi_0(G)$ -sets. If it exists, the arrow $\overline{\pi_0(q)}$ is unique which implies naturality.

Denote path-components by $[\cdot]$. In order to show that $\overline{\pi_0(q)}$ exists, pick $[x] \in \pi_0(X)$. Then for all $[g] \in \pi_0(G)$, we have $\pi_0(q)([x]) = [q(x)] = [q(xg)] = \pi_0(q)([xg]) = \pi_0(q)([x] \cdot [g])$ where \cdot denotes the $\pi_0(G)$ -action on $\pi_0(X)$. Thus, $\pi_0(q)$ factors through the quotient map ϱ .

If $X = \emptyset$, then both domain and codomain of $\overline{\pi_0(q)}$ are one-point sets and the map is an isomorphism. Otherwise, the quotient map $q : X \rightarrow X/G$ is surjective and induces a surjection on π_0 . Consequently, the map $\overline{\pi_0(q)}$ is surjective as well, and it remains to show that the map is injective.

Assume that for $x, y \in X$, we have $\overline{\pi_0(q)}(\varrho([x])) = \overline{\pi_0(q)}(\varrho([y]))$, i.e., $\pi_0(q)([x]) = \pi_0(q)([y])$, which can be rephrased as $[q(x)] = [q(y)]$. This means that there is a path $\gamma : [0; 1] \rightarrow X/G$ from $q(x)$ to $q(y)$. As the quotient map $X \rightarrow X/G$ is a fibration, we can lift γ to a path $\tilde{\gamma} : [0; 1] \rightarrow X$ from $\tilde{\gamma}(0) = x$ to some point $\tilde{\gamma}(1)$ with $q(\tilde{\gamma}(1)) = q(y)$. This implies that there is $g \in G$ with $\tilde{\gamma}(1) = yg$. In particular, we have

$$\varrho([x]) = \varrho([\tilde{\gamma}(1)]) = \varrho([yg]) = \varrho([y] \cdot [g]) = \varrho([y]) \in \pi_0(X)/\pi_0(G),$$

and we deduce that $\overline{\pi_0(q)}$ is injective. \square

Definition A.13. Let X and G be as before. Assume additionally that $X \neq \emptyset$ and that the G -action on X is free. Picking $x \in X$, denote by $\text{fib}_{q(x)}(q)$ the fiber of $q : X \rightarrow X/G$ over $q(x) \in X/G$. We equip this fiber with a group structure using the homeomorphism

$$G \xrightarrow{\cong} \text{fib}_{q(x)}(q), \quad g \mapsto xg^{-1}.$$

Remark A.14. The assignment $g \mapsto xg$ is a homeomorphism as well and could have been used to define the group structure. However, the convention described above ensures that the group structure is compatible with the standard conventions for homotopy groups. Moreover, note that this group structure is natural with respect to G -equivariant pointed maps of pointed free G -spaces.

Proposition A.15. *Let X , x , and G be as in Definition A.13. The boundary map*

$$\pi_1(X/G, q(x)) \xrightarrow{\partial} \pi_0(\text{fib}_{q(x)}(q))$$

associated to the fibration $q : X \rightarrow X/G$ is a group homomorphism with respect to the group structure which is induced by the group structure from Definition A.13 under the functor π_0 .

Proof. We interpret $\pi_1(X/G, q(x))$ as classes of maps of pairs of spaces $(D^1, S^0) \rightarrow (X/G, q(x))$ and employ the notation $[\cdot]$ for classes in π_0 or π_1 . Let α_1 be such a map. As q is a fibration, we may find a lift $\widetilde{\alpha}_1 : D^1 \rightarrow X$ such that $q \circ \widetilde{\alpha}_1 = \alpha_1$ and $\widetilde{\alpha}_1(0) = x$. Then $\partial[\alpha_1] = [\widetilde{\alpha}_1(1)] \in \text{fib}_{q(x)}(q)$. There is a unique $g \in G$ such that $\widetilde{\alpha}_1(1) = xg^{-1}$.

Given another $\alpha_2 : (D^1, S^0) \rightarrow (X/G, q(x))$, choose a lift $\widetilde{\alpha}_2$ as before such that $\partial[\alpha_2] = [\widetilde{\alpha}_2(1)]$. Let $h \in G$ be the unique element with $\widetilde{\alpha}_2(1) = xh^{-1}$. We wish to show that $\partial([\alpha_1] \cdot [\alpha_2])$ agrees with $\partial([\alpha_1]) \cdot \partial([\alpha_2]) = [xg^{-1}] \cdot [xh^{-1}]$ which is $[x(gh)^{-1}] = [xh^{-1}g^{-1}]$ by definition of the group structure on $\pi_0(\text{fib}_{q(x)}(q))$.

The product $[\alpha_1] \cdot [\alpha_2]$ in $\pi_1(X, x)$ is represented by the concatenation $\alpha_1 * \alpha_2$ with the standard convention $(\alpha_1 * \alpha_2)(0) = \alpha_1(0)$, $(\alpha_1 * \alpha_2)(1) = \alpha_2(1)$. Denote by $\widetilde{\alpha}_2 \cdot g^{-1}$ the path $t \mapsto \widetilde{\alpha}_2(t) \cdot g^{-1}$ from xg^{-1} to $xh^{-1}g^{-1}$. Then $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2 \cdot g^{-1}$ are concatenable and their concatenation $\gamma := \widetilde{\alpha}_1 * (\widetilde{\alpha}_2 \cdot g^{-1})$ satisfies $q \circ \gamma = \alpha_1 * \alpha_2$ and $\gamma(0) = x$. Hence, $\partial([\alpha_1] \cdot [\alpha_2]) = \partial([\alpha_1 * \alpha_2]) = [\gamma(1)] = [xh^{-1}g^{-1}]$ concluding the proof. \square

Proof of Theorem A.8. In the trivial case $X = \emptyset$, the statement holds. Let us exclude this case from the following considerations.

Observe that X and EG are Hausdorff by assumption and by Lemma A.7, respectively. Therefore, their product $X \times_0 EG$ in the category of general topological spaces is Hausdorff. As all open sets in $X \times_0 EG$ are also open in its k -ification $X \times EG = k(X \times_0 EG)$, we deduce that $X \times EG$ is Hausdorff as well.

Denote the projection $X \times EG \rightarrow X$ by pr and pick $(x, e) \in X \times EG$. We obtain a commutative diagram of fiber sequences:

$$\begin{array}{ccccc} \text{fib}_{q_1(x,e)}(q_1) & \longrightarrow & X \times EG & \xrightarrow{q_1} & X \times_G EG \\ \text{fib}(\text{pr}, \text{pr}/G) \downarrow & & \downarrow \text{pr} & & \downarrow \text{pr}/G \\ \text{fib}_{q_2(x)}(q_2) & \longrightarrow & X & \xrightarrow{q_2} & X/G \end{array}$$

Using homeomorphisms as in Definition A.13, we obtain a commutative triangle

$$\begin{array}{ccc} & \text{fib}_{q_1(x,e)}(q_1) = (x, e)G & \\ \cong \nearrow & \downarrow \text{fib}(\text{pr}, \text{pr}/G) & \\ G/\{1\} \cong G & & \\ \cong \searrow & \downarrow & \\ & \text{fib}_{q_2(x)}(q_2) = xG & \end{array}$$

Thus, $\text{fib}(\text{pr}, \text{pr}/G)$ is a homeomorphism. Since EG is contractible, $\text{pr} : X \times EG \rightarrow X$ is a weak equivalence. By Theorem A.10, the quotient maps q_1 and q_2 are fibrations. In the associated long exact sequences on π_* , the terms $\pi_0(\text{fib}_{q_1(x,e)}(q_1))$ and $\pi_0(\text{fib}_{q_2(x)}(q_2))$ inherit group structures, and the appropriate boundary maps are group homomorphisms, see Definition A.13 and Proposition A.15. This is sufficient to deduce that pr/G induces an isomorphism on π_n for $n \geq 1$ and all basepoints by the four lemmas.

Finally, the weak equivalence $\text{pr} : X \times EG \rightarrow X$ is G -equivariant and induces an isomorphism of $\pi_0(G)$ -sets on π_0 . Thus, it induces an isomorphism $\pi_0(\text{pr})/\pi_0(G) : \pi_0(X \times EG)/\pi_0(G) \cong \pi_0(X)/\pi_0(G)$. The diagram

$$\begin{array}{ccc} \pi_0(X \times EG)/\pi_0(G) & \xrightarrow[\cong]{\pi_0(\text{pr})/\pi_0(G)} & \pi_0(X)/\pi_0(G) \\ \pi_0(q_1) \downarrow \cong & & \cong \downarrow \pi_0(q_2) \\ \pi_0(X \times_G EG) & \xrightarrow{\pi_0(\text{pr}/G)} & \pi_0(X/G) \end{array}$$

commutes, and the vertical arrows are isomorphisms, see Proposition A.12. Therefore, the map pr/G is an isomorphism on π_0 , too, and it is a weak equivalence. \square

A.5. The Topology of $\tilde{E}(H, G)$

In order to apply Theorem A.8 to $\tilde{E}(H, G)$, which is necessary for the proof of Proposition 2.22, we need to verify that this is a Hausdorff space. Let us start by recalling the topology on $E(H, G)$.

Observation A.16. *The underlying space of the $H \times G$ -space $E(H, G)$ is just $\mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$. Now, recall from [12, Section 2] that given finite-dimensional real inner product spaces V, W , the space $\mathbb{L}(V, W)$ of linear isometric embeddings is topologized by choosing an orthonormal basis (v_1, \dots, v_k) of V and using the bijection*

$$\begin{array}{ccc} \mathbb{L}(V, W) & \longrightarrow & \text{Stiefel}_k(W) \\ f & \mapsto & (f(v_1), \dots, f(v_k)) \end{array}$$

to the Stiefel manifold of k -frames in W . This Stiefel manifold is a subset of W^k and endowed with the subspace topology.

Having established the topology on $\mathbb{L}(V, W)$ for V and W finite-dimensional, we move on to the case where the second argument is of countably infinite dimension. For such a \mathcal{W} , we define

$$\mathbb{L}(V, \mathcal{W}) := \operatorname{colim}_{W \in s(\mathcal{W})} \mathbb{L}(V, W)$$

where $s(\mathcal{W})$ is the poset of finite-dimensional subspaces of \mathcal{W} . For $W \leq W' \in s(\mathcal{W})$, the induced map $\mathbb{L}(V, W) \longrightarrow \mathbb{L}(V, W')$ is a closed inclusion. Furthermore, If $W_0 \subseteq W_1 \subseteq \dots \subseteq \mathcal{W}$ is a strictly increasing sequence of finite-dimensional subspaces such that $\cup_i W_i = \mathcal{W}$, it is cofinal, and we have

$$\mathbb{L}(V, \mathcal{W}) = \operatorname{colim}_i \mathbb{L}(V, W_i).$$

As this colimit is taken along closed inclusions, it agrees with the colimit computed in general topological spaces (Lemma A.1).

It remains to upgrade the first variable to vector spaces of countably infinite dimension. For such a vector space \mathcal{V} , we set

$$\mathbb{L}(\mathcal{V}, \mathcal{W}) = \lim_{V \in s(\mathcal{V})} \mathbb{L}(V, \mathcal{W})$$

where the limit is taken in the category of CGWH spaces. Again, for an exhaustive sequence $V_0 \subseteq V_1 \subseteq \dots$ of finite dimensional subspaces of \mathcal{V} , we have

$$\mathbb{L}(\mathcal{V}, \mathcal{W}) = \lim_i \mathbb{L}(V_i, \mathcal{W}).$$

Lemma A.17. *The space $\mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is Hausdorff.*

Proof. First, the spaces $\mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)$ are normal as they are metrizable. Therefore, $\mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$, which is defined as $\operatorname{colim}_m \mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)$, is a sequential colimit of normal spaces along closed inclusions. Thus, $\mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ is normal by Lemma A.6. The space $\mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is T_1 , too, so it is Hausdorff.

The limit $\mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) = \lim_n \mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ can be modeled as a (closed) subspace of the infinite product $\prod_n \mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ of Hausdorff spaces. As subspaces of Hausdorff spaces are Hausdorff, it suffices to show that $\prod_n \mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ is Hausdorff. This is true for the product computed in general topological spaces. The product in the category of CGWH spaces can then be obtained by k -ification. However, this procedure does not destroy Hausdorffness because k -ification preserves open sets. We conclude that $\prod_n \mathbb{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ and its subspace $\mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ are Hausdorff. \square

We are now ready to obtain a few important facts about the topology on $\tilde{E}(H, G)$.

Lemma A.18. *The two topologies $\operatorname{map}(H, G) \times E(H, G)$ and $\operatorname{map}(H, G) \times_0 E(H, G)$ agree, and they are Hausdorff. The subspace $\tilde{E}(H, G)$ is compactly generated and Hausdorff. In particular, it is CGWH.*

Proof. The space $\operatorname{map}(H, G)$ is metrizable and hence locally compact Hausdorff. Therefore, the product $\operatorname{map}(H, G) \times E(H, G)$ agrees with the product $\operatorname{map}(H, G) \times_0 E(H, G)$ taken in general topological spaces.

For the second statement, first observe that $\operatorname{map}(H, G)$ is Hausdorff because of its metrizability and that the space $E(H, G)$ is Hausdorff by Lemma A.17. Therefore, $\operatorname{map}(H, G) \times E(H, G)$ is Hausdorff and so is its subspace $\tilde{E}(H, G)$. This implies the weak Hausdorff property of course.

To prove that $\tilde{E}(H, G)$ is compactly generated, recall that the map $\tilde{p}_{H, G} : \tilde{E}(H, G) \rightarrow \text{map}(H, G)$ is a fiber bundle by Proposition 2.15. In particular, the proof shows that each point of $\tilde{E}(H, G)$ has a neighborhood that is homeomorphic to $U \times E(H, G)^\alpha$ for $U \subseteq \text{map}(H, G)$ open and $\alpha \in \text{map}(H, G)$.

According to our convention, the product $U \times E(H, G)^\alpha$ is computed in the category of CGWH spaces, and therefore automatically compactly generated. As U is locally compact Hausdorff, we actually have $U \times \text{map}(H, G) = U \times_0 \text{map}(H, G)$.

In any case, the space $\tilde{E}(H, G)$ is locally compactly generated which implies that it is compactly generated, concluding the proof. \square

A.6. Cofibrant Objects in Cofibrantly Generated Model Categories

Recall that cofibrations in a cofibrantly generated model category \mathcal{M} with generating cofibrations I are precisely the retracts of transfinite compositions of pushouts of elements of I [5, Corollary 10.5.22]. Under an additional assumption, which is satisfied by $\text{Pre}(\mathcal{C}, \text{Top})$, this allows us to derive a characterization of cofibrant objects.

Lemma A.19. *Let \mathcal{M} be as above and suppose that any map $X \rightarrow \emptyset$ is an isomorphism. Let \mathcal{N} be a class of objects of \mathcal{M} . Assume that \mathcal{N} satisfies the following properties:*

- (i) \mathcal{N} is closed under retracts.
- (ii) If $X \in \mathcal{N}$ and X' is a pushout of X along a generating cofibration,

$$\begin{array}{ccc} A & \longrightarrow & X \\ I \ni \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & X' \end{array}$$

then $X' \in \mathcal{N}$.

- (iii) If λ is an ordinal and $X : \lambda \rightarrow \mathcal{M}$ is a λ -sequence such that

- a) $X_0 = \emptyset$ if $\lambda > 0$,
- b) $X_\beta \in \mathcal{N}$ for $\beta < \lambda$, and
- c) for $\beta < \lambda$ such that $\beta + 1 < \lambda$, there is a pushout

$$\begin{array}{ccc} A_\beta & \longrightarrow & X_\beta \\ I \ni \downarrow & \lrcorner & \downarrow \\ B_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

then $\text{colim}_{\beta < \lambda} X_\beta \in \mathcal{N}$.

Then \mathcal{N} contains all cofibrant objects. Moreover, the class of cofibrant objects is the smallest class satisfying these properties.

Proof. First, note that $\emptyset \in \mathcal{N}$ because the empty 0-sequence, whose colimit is \emptyset , satisfies all the conditions (iii).(a)-(c).

Let $X' \in \mathcal{M}$ be cofibrant, i.e., the map $\emptyset \rightarrow X'$ is a cofibration. We wish to show that $X' \in \mathcal{N}$. As \mathcal{M} is cofibrantly generated, this means that there is an ordinal λ and a λ -sequence $X : \lambda \rightarrow \mathcal{M}$ such that

- there is a retract diagram

$$\begin{array}{ccccc} & & \text{id} & & \\ & \nearrow & & \searrow & \\ \emptyset & \longrightarrow & X_0 & \longrightarrow & \emptyset \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & \text{colim}_{\beta < \lambda} X_\beta & \longrightarrow & X' \\ & \searrow & & \nearrow & \\ & & \text{id} & & \end{array}$$

- for $\beta < \lambda$ such that $\beta + 1 < \lambda$, there is a pushout

$$\begin{array}{ccc} A_\beta & \longrightarrow & X_\beta \\ I \ni \downarrow & & \downarrow \\ B_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

As the map $X_0 \longrightarrow \emptyset$ is an isomorphism by assumption, we can assume without loss of generality that $X_0 = \emptyset$. Furthermore, \mathcal{N} is closed under retracts, therefore, it suffices to show that $\text{colim}_{\beta < \lambda} X_\beta \in \mathcal{N}$.

Conditions (iii).(a) and (iii).(c) hold for the λ -sequence X , and we wish to verify the remaining condition (iii).(b), i.e., $X_\beta \in \mathcal{N}$ for $\beta < \lambda$. If this is not the case, let $\mu < \lambda$ be the smallest ordinal such that $X_\mu \notin \mathcal{N}$. Since $\emptyset \in \mathcal{N}$ and $X_0 = \emptyset$, we must have $\mu > 0$. The truncated diagram $X|_\mu$ is a μ -sequence and satisfies (iii).(a)-(c). Hence, $\text{colim}_{\beta < \mu} X_\beta \in \mathcal{N}$.

If $\mu = \nu + 1$ is a successor ordinal, we obtain $\text{colim}_{\beta < \mu} X_\beta = X_\nu \in \mathcal{N}$. As X_μ is a pushout of X_ν along a generating cofibration, $X_\mu \in \mathcal{N}$ by (ii), a contradiction.

If μ is a limit ordinal, we immediately obtain $\text{colim}_{\beta < \mu} X_\beta = X_\mu$ because X is a λ -sequence. Therefore, $X_\mu \in \mathcal{N}$, a contradiction as well. In conclusion, $\text{colim}_{\beta < \lambda} X_\beta \in \mathcal{N}$ by (iii), and we have shown that \mathcal{N} contains all cofibrant objects.

For the second claim, we need to check that the class \mathcal{M}^{cof} of cofibrant objects of \mathcal{M} satisfies (i)-(iii). This is obvious for (i) and (ii). If X is a λ -sequence such that (iii).(a)-(c) hold (note that (b) actually follows from (a) and (c) in this case), then the map $\emptyset = X_0 \longrightarrow \text{colim}_{\beta < \lambda} X_\beta$ is a cofibration, and the object $\text{colim}_{\beta < \lambda} X_\beta$ is cofibrant. \square

A.7. Induced Equivalences on Enriched Presheaves

Having established a zig-zag of Dwyer-Kan equivalences between indexing categories in Corollary 3.13, we would like to deduce that the associated enriched presheaf categories are Quillen equivalent. A short proof of this result can be found in [4, Lemma A.6]. In the following, we will present a more detailed version. Before we get started, let us record some general properties of enriched presheaves and introduce notation.

Definition A.20. Let \mathcal{C} be a topologically enriched category.

- (i) Denote by $\text{Pre}(\mathcal{C}, \text{Top})$ the category of enriched functors $\mathcal{C} \longrightarrow \text{Top}$. In this paper, it will always be equipped with the projective model structure.

- (ii) The category $\text{Pre}(\mathcal{C}, \text{Top})$ is bitensored over Top . For $X \in \text{Pre}(\mathcal{C}, \text{Top})$, $K \in \text{Top}$, and $c \in \mathcal{C}$, we have

$$(X \otimes K)(c) = X(c) \times K, \quad (X^K)(c) = X(c)^K.$$

- (iii) For every $c \in \mathcal{C}$, write $\text{ev}_c : \text{Pre}(\mathcal{C}, \text{Top}) \longrightarrow \text{Top}$ for the functor given by evaluation at c .

Proposition A.21. Let \mathcal{C} be a topologically enriched category.

- (i) For $c \in \mathcal{C}$, the functor ev_c has a left adjoint $F_c : \text{Top} \longrightarrow \text{Pre}(\mathcal{C}, \text{Top})$, given by $(F_c K)(c') = \mathcal{C}(c', c) \times K$. The pair $F_c \dashv \text{ev}_c$ is Quillen.

- (ii) The projective model structure on $\text{Pre}(\mathcal{C}, \text{Top})$ is cofibrantly generated by the generating cofibrations

$$I_{\mathcal{C}} := \{F_c(S^{n-1} \hookrightarrow D^n) \mid n \geq 0, c \in \mathcal{C}\}$$

and the generating trivial cofibrations

$$J_{\mathcal{C}} := \{F_c(D^n \hookrightarrow D^n \times [0; 1]) \mid n \geq 0, c \in \mathcal{C}\}.$$

The domains and codomains of the maps in $I_{\mathcal{C}} \cup J_{\mathcal{C}}$ are cofibrant.

- (iii) $\text{Pre}(\mathcal{C}, \text{Top})$ is a topological model category. In particular, we have natural homeomorphisms

$$\text{Pre}(\mathcal{C}, \text{Top})(X \otimes K, X') \cong \text{Pre}(\mathcal{C}, \text{Top})(X, (X')^K)$$

and the pushout-product axiom holds.

(iv) Any cofibration in $\text{Pre}(\mathcal{C}, \text{Top})$ is a levelwise closed cofibration, i.e., a Hurewicz cofibration with closed image.

Proof. It is an easy exercise to verify that F_c is left adjoint to ev_c . As ev_c preserves fibrations and trivial fibrations, the pair $F_c \dashv \text{ev}_c$ is Quillen, showing (i).

The category of (CGWH) topological spaces satisfies the monoid axiom by [6, Lemma 2.3]. Therefore, we may apply [14, Theorem 24.4] and deduce (ii). Part (iii) is a straight-forward computation.

Part (iv) uses the theory of *h-cofibrations*. see, e.g., [11, Definition A.1.15]. Every topological space is fibrant, so every object of $\text{Pre}(\mathcal{C}, \text{Top})$ is fibrant. From [11, Corollary A.1.17.(iii)], we deduce that every cofibration in $\text{Pre}(\mathcal{C}, \text{Top})$ is a h-cofibration. Picking $c \in \mathcal{C}$, the functor $\text{ev}_c : \text{Pre}(\mathcal{C}, \text{Top}) \rightarrow \text{Top}$ commutes with colimits and tensors. Thus, it takes h-cofibrations to cofibrations by part (ii) of the same Corollary. So, every cofibration is a levelwise h-cofibration of topological spaces. An inspection of the definitions shows that h-cofibrations in Top are exactly the Hurewicz cofibrations.

To conclude the proof of part (iv), it remains to show that a cofibration in $\text{Pre}(\mathcal{C}, \text{Top})$ is levelwise a closed inclusion. This is true for the generating cofibrations by their description above. The characterization of cofibrations, see Subsection A.6, together with Lemma A.2 shows that it is also true for an arbitrary cofibration. \square

Proposition A.22. *Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be an enriched functor between topologically enriched categories. Then the restriction functor $f^* : \text{Pre}(\mathcal{D}, \text{Top}) \rightarrow \text{Pre}(\mathcal{C}, \text{Top})$ has a left adjoint $f_!$. The pair $f_! \dashv f^*$ is Quillen. Furthermore, both functors respect the tensoring over Top . Finally, both functors preserve colimits.*

Proof. The construction of $f_!$ can be found in [4, Lemma A.6]. Alternatively, one can use the left Kan extension of the functor

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \text{Pre}(\mathcal{D}, \text{Top}) \\ c & \mapsto & F_{f(c)}(*) \end{array}$$

along the enriched Yoneda embedding $F_-(*) : \mathcal{C} \rightarrow \text{Pre}(\mathcal{C}, \text{Top})$. To show that the pair $f_! \dashv f^*$ is Quillen, it suffices to observe that f^* clearly preserves fibrations and trivial fibrations.

The left adjoint $f_!$ preserves colimits, and f^* preserves colimits, too, because they are computed levelwise. Moreover, it is evident from the definitions that f^* preserves the tensoring. To show that $f_!$ preserves the tensoring, observe that the cotensoring is also compatible with f^* . Let $X \in \text{Pre}(\mathcal{C}, \text{Top})$, $Y \in \text{Pre}(\mathcal{D}, \text{Top})$, and $K \in \text{Top}$. We compute

$$\begin{aligned} & \text{Pre}(\mathcal{D}, \text{Top})(f_!(X \otimes K), Y) \\ & \cong \text{Pre}(\mathcal{C}, \text{Top})(X \otimes K, f^*Y) \\ & \cong \text{Pre}(\mathcal{C}, \text{Top})(X, (f^*Y)^K) \\ & \cong \text{Pre}(\mathcal{C}, \text{Top})(X, f^*(Y^K)) \\ & \cong \text{Pre}(\mathcal{D}, \text{Top})(f_!X, Y^K) \\ & \cong \text{Pre}(\mathcal{D}, \text{Top})((f_!X) \otimes K, Y) \end{aligned}$$

naturally. Hence, there is a natural isomorphism $f_!(X \otimes K) \cong (f_!X) \otimes K$. \square

The following two lemmas provide the technical details necessary for the proof of the main theorem at the end of this subsection.

Lemma A.23. *Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be as before. If the functor f is homotopically fully faithful, then the unit $\eta_X : X \rightarrow f^*f_!X$ is a level weak equivalence in $\text{Pre}(\mathcal{C}, \text{Top})$ for all cofibrant $X \in \mathcal{C}$.*

Proof. Let \mathcal{N} be the class of all objects of $\text{Pre}(\mathcal{C}, \text{Top})$ for which the unit is a weak equivalence. Our goal is to show that the conditions of Lemma A.19 are satisfied.

The only map whose codomain is the empty presheaf \emptyset is its identity id_\emptyset , which is an isomorphism. We proceed to verify (i)-(iii).

- (i) If $X \in \mathcal{N}$ and X' is a retract of X , then the unit map $\eta_{X'}$ is a retract of η_X by naturality of the unit. A retract of a weak equivalence is a weak equivalence. Hence, $\eta_{X'}$ is a weak equivalence.

- (ii) First, let us show that η_X is a weak equivalence whenever X is the domain or codomain of a map in I . These domains and codomains are of the form $F_c(K) = F_c(*) \otimes K$ for $c \in \mathcal{C}$ and K a space. Evaluating $\eta_{F_c(K)}$ at some $c' \in \mathcal{C}$ yields

$$\begin{array}{ccc}
 (F_c(*) \otimes K)(c') & \xrightarrow{\eta_{F_c(K)}} & (f^* f_! (F_c(*) \otimes K))(c') \\
 \parallel & & \parallel \\
 \mathcal{C}(c', c) \times K & & (f_! (F_c(*) \otimes K))(f(c')) \\
 & & \downarrow \cong \\
 & & (f_! (F_c(*)) \otimes K)(f(c')) \\
 & & \downarrow \cong \\
 & & (F_{f(c)}(*) \otimes K)(f(c')) \\
 & & \parallel \\
 & & \mathcal{D}(f(c'), f(c)) \times K
 \end{array}$$

$f_{c',c} \times K$

using the compatibility of $f_!$ with tensors and the formula $f_!(F_c(*)) \cong F_{f(c)}(*)$ from the proof of Proposition A.22. The bent map is a weak equivalence because $f_{c',c} : \mathcal{C}(c', c) \rightarrow \mathcal{D}(f(c'), f(c))$ is a weak equivalence since f is homotopically fully faithful by assumption. Hence $\eta_{F_c(K)}$ is a level weak equivalence.

Given a pushout diagram

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 I \Downarrow & & \downarrow \\
 B & \longrightarrow & X'
 \end{array}$$

with $X \in \mathcal{N}$, the square

$$\begin{array}{ccc}
 f^* f_! A & \longrightarrow & f^* f_! X \\
 \downarrow & & \downarrow \\
 f^* f_! B & \longrightarrow & f^* f_! X'
 \end{array}$$

is pushout by Proposition A.22. Evaluating at some $c \in \mathcal{C}$, we obtain a commutative cube

$$\begin{array}{ccccc}
 & (f^* f_! A)(c) & \longrightarrow & (f^* f_! X)(c) & \\
 & \nearrow \simeq & & \nearrow \simeq & \\
 A(c) & \longrightarrow & X(c) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & (f^* f_! B)(c) & \longrightarrow & (f^* f_! X')(c) & \\
 \nearrow \simeq & & \nearrow \simeq & & \\
 B(c) & \longrightarrow & X'(c) & &
 \end{array}$$

with both the front and rear faces being pushout. The maps perpendicular to the plane of projection are the respective unit maps evaluated at c . Three of these are weak equivalences, namely $\eta_X(c)$ by assumption and $\eta_A(c)$ and $\eta_B(c)$ by our considerations from before.

The map $A \rightarrow B$ is a cofibration, so $A(c) \rightarrow B(c)$ is a Hurewicz cofibration by Proposition A.21.(iv). The left adjoint $f_!$ preserves cofibrations, so $f_! A \rightarrow f_! B$ is a cofibration and

$$(f^* f_! A)(c) = (f_! A)(f(c)) \rightarrow (f_! B)(f(c)) = (f^* f_! B)(c)$$

is a Hurewicz cofibration. In this situation, the map $\eta_{X'}(c)$ between the pushouts is a weak equivalence as well ([1, Proposition 4.8 (b)] or [8, Proposition 1.1]). Since c was arbitrary, $\eta_{X'}$ is a level weak equivalence.

(iii) Assume that there is a λ -sequence $X : \lambda \longrightarrow \text{Pre}(\mathcal{C}, \text{Top})$ such that (a)-(c) from Lemma A.19 hold. A case distinction is in order.

λ is 0 A 0-sequence is just an empty diagram with colimit the initial object \emptyset . Also, $f^*f_!\emptyset = \emptyset$ and the map η_\emptyset must be id_\emptyset which is a weak equivalence.

λ a successor ordinal If $\lambda = \mu + 1$, we have $\text{colim}_{\beta < \lambda} X_\beta = X_\mu \in \mathcal{N}$ by assumption.

λ a limit ordinal Some categorical yoga shows that $\text{colim}_{\beta < \lambda} \eta_{X_\beta} = \eta_{\text{colim}_{\beta < \lambda} X_\beta}$. Therefore, it is sufficient to show that $\text{colim} \eta_{X_\beta}$ is a weak equivalence.

By condition (b), the maps $X_\beta \longrightarrow X_{\beta+1}$ are cofibrations, hence levelwise closed inclusions by Proposition A.21. The maps $f_!X_\beta \longrightarrow f_!X_{\beta+1}$ are cofibrations and levelwise closed inclusions as well, and we derive that $f^*f_!X_\beta \longrightarrow f^*f_!X_{\beta+1}$ are levelwise closed inclusions. Hence, both $\text{colim} X_\beta(c)$ and $\text{colim} f^*f_!X_\beta(c)$ are taken along closed inclusions, and the maps $X_\beta \longrightarrow f^*f_!X_\beta$ are weak equivalences because $X_\beta \in \mathcal{N}$ by assumption (c):

$$\begin{array}{ccc} X_\beta(c) & \xrightarrow{\simeq} & (f^*f_!X_\beta)(c) \\ \downarrow & & \downarrow \\ X_{\beta+1}(c) & \xrightarrow{\simeq} & (f^*f_!X_{\beta+1})(c) \\ \vdots & & \vdots \\ \text{colim } X_\beta(c) & \xrightarrow{\text{colim } \eta_{X_\beta}} & \text{colim } (f^*f_!X_\beta)(c) \end{array}$$

By Lemma A.3, $\pi_k \text{colim } X_\beta(c) \cong \text{colim } \pi_k X_\beta(c)$ and $\pi_k \text{colim } (f^*f_!X_\beta)(c) \cong \text{colim } \pi_k (f^*f_!X_\beta)(c)$. Thus, $\pi_k(\text{colim } \eta_{X_\beta})$ is a colimit of isomorphisms and therefore an isomorphism itself. In conclusion, $\eta_{\text{colim } X_\beta} = \text{colim } \eta_{X_\beta}$ is a weak equivalence as desired. Note that we did not need condition (a) in our specific situation. \square

Lemma A.24. *Let \mathcal{D} be a topologically enriched category and $Y \in \text{Pre}(\mathcal{D}, \text{Top})$. If $a : d \longrightarrow d'$ is a homotopy equivalence in \mathcal{D} , then so is $Y(a) : Y(d) \longrightarrow Y(d')$.*

Proof. If a is such a homotopy equivalence, then there is $b : d' \longrightarrow d \in \mathcal{D}$ such that $[b \circ a] = [\text{id}_d] \in \pi_0(\mathcal{D}(d, d))$ and $[a \circ b] = [\text{id}_{d'}] \in \pi_0(\mathcal{D}(d', d'))$.

The enriched functor Y induces a map

$$\pi_0(\mathcal{D}(d, d)) \longrightarrow \pi_0(\text{Top}(Y(d), Y(d))).$$

As $[b \circ a]$ and $[\text{id}_d]$ are the same on the left hand side, we obtain that $[Y(\text{id}_d)] = [\text{id}_{Y(d)}]$ and $[Y(b \circ a)] = [Y(a) \circ Y(b)]$ agree, too. Two maps represent the same path component in $\pi_0(\text{Top}(Y(d), Y(d)))$ if and only if they are homotopic. Therefore, $Y(a) \circ Y(b) \simeq \text{id}_{Y(d)}$. By an analogous argument, $Y(b) \circ Y(a) \simeq \text{id}_{Y(d')}$. We conclude that $Y(a)$ is a homotopy equivalence. \square

Theorem A.25. *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a Dwyer-Kan equivalence between topologically enriched categories. Then the induced Quillen pair $f_! \dashv f^*$ is a Quillen equivalence.*

Proof. Let $X \in \text{Pre}(\mathcal{C}, \text{Top})$ be cofibrant, $Y \in \text{Pre}(\mathcal{D}, \text{Top})$, and $\alpha : X \longrightarrow f^*Y$ be a map with adjoint $\beta : f_!X \longrightarrow Y$. By one of the many equivalent characterizations of a Quillen equivalence, we need to show that α is a (level) weak equivalence if and only β is (note that any Y is fibrant). The map α factors through the unit η_X , which is a weak equivalence by Lemma A.23:

$$\begin{array}{ccc} & f^*f_!X & \\ \eta_X \nearrow & & \searrow f^*(\beta) \\ X & \xrightarrow{\alpha} & f^*Y \end{array}$$

Therefore, α is a weak equivalence if and only if $f^*(\beta)$ is. It remains to show that this is the case if and only if β is a weak equivalence.

Assume that $f^*(\beta)$ is a weak equivalence and choose $d \in \mathcal{D}$. As f is homotopically essentially surjective, there is $c \in \mathcal{C}$ and a map $a : fc \rightarrow d$ in \mathcal{D} that is a homotopy equivalence. We obtain a commutative diagram

$$\begin{array}{ccc} (f_!X)(fc) = (f^*f_!X)(c) & \xrightarrow[\simeq]{\beta_{fc}=(f^*(\beta))_c} & Y(fc) = (f^*Y)(c) \\ \uparrow a^* & & \uparrow a^* \\ (f_!X)(d) & \xrightarrow{\beta_d} & Y(d) \end{array}$$

The vertical arrows are homotopy equivalences by Lemma A.24, therefore, β_d is a weak equivalence and β is a level weak equivalence.

Vice versa, let β be a level weak equivalence. Obviously, this implies that $f^*(\beta)$ is a level weak equivalence, concluding the proof. \square

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